DAILY UPDATE FOR MATH 147 FALL 2024

Monday, August 26. This class was devoted to an informal discussion of the topics we plan to cover this semester. We began by describing the type of functions we will study, namely scalar valued and vector valued functions of several variables. We illustrated, in a general way, how with the proper definitions, many of the basic concepts from Calculus I and Calculus II can be extended to functions of several variables. On the other hand, we noted, for example, that for a function of two variables, one can measure a rate of change in infinitely many different directions, something not encountered in Calculus I.

We then gave a brief overview of how the integration process works in general: One always has a function to integrate (the *integrand*) and a domain of integration. We then described how the integration process works in all scenarios we will encounter during the semester. Namely, starting with a domain of integration and a function defined on that domain, we proceed as follows:

(i) Subdivide the domain of integration into small portions of a similar type, e.g, if the domain of integration is a solid, subdivide into smaller solids; if the domain of integration is a curve, subdivide into smaller curves. One can assume the individual portions have the same size.
(ii) Choose a point in each subdivision and evaluate the function at that point.
(iii) Multiply the answer in (ii) by the size of the subdivision, e.g., volume if a solid, length if a curve.
(iv) Add the quantities in (iii).
(v) Take the limit of the sums in (iv) as the size of the subdivisions tend to zero.

The resulting numerical value depends only on the function and the underlying geometry of the domain of integration. We noted that the real challenge is to calculate this quantity.

We continued our discussion of functions of several variables, with the examples (like, though not exactly) $f(x,y) = x^2 + y^2$, $g(x,y,z) = \frac{1}{x^2+y^2+z^2}$, and $h(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$. We noted that $f(-2,3) = (-2)^2 + 3^2 = 13$, $g(1,2,3) = \frac{1}{1^2+2^2+3^2} = \frac{1}{14}$, and $h(1,2,\ldots,n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

We then noted that the range and domain for functions of several variables, have the same meaning as for functions of a single variable, namely the domain is the set of allowable inputs and the range is the set of possible outputs. In the case of a function of two variables, say, the allowable inputs will elements of $\mathbb{R}^2$, but the the outputs are always real numbers. For the functions defined above, we have:

(i) $f(x,y)$: Domain $= \mathbb{R}^2$ and range $= \{\alpha \in \mathbb{R} \mid \alpha \geq 0\}$.
(ii) $g(x,y,z)$: Domain $\mathbb{R}^3\backslash(0,0,0)$ and range $= \{\alpha \in \mathbb{R} \mid \alpha > 0\}$.
(iii) $h(x_1, x_2, \ldots, x_n)$: Domain $= \mathbb{R}^n$ and range $= \mathbb{R}$.

One notes that for the function $t(x,y) = \sqrt{1 - x^2 - y^2}$, the domain is the the unit circle of radius one centered at the origin in $\mathbb{R}^2$ and its interior, while the range is $\{\alpha \in \mathbb{R} \mid 0 \leq \alpha \leq 1\}$. After this we discussed graphing functions of two variables $z = f(x,y)$ and how the level curves $f(x,y) = c$ for different values of c help to understand the graph of $f(x,y)$. We also cautioned that the level curves do not give a complete picture of the graph, since for example, the level curves of $f(x,y) = x^2 + y^2$ and $g(x,y) = \sqrt{x^2+y^2}$ are circles of increasing radii, as $c > 0$ increases, but the graph of $f(x,y)$ is a *paraboloid* (see here) while the graph of $g(x,y)$ is a *cone* (see here). We noted that the difference between these two surfaces can be seen by taking curves obtained by setting $x = c$, and in particular $x = 0$. In this cross section, the graph of $f(x,y)$ is a parabola while the graph of $g(x,y)$ is the graph of $z = |y|$ in the yz-plane, i.e., two rays emanating from the origin at 45 degrees.

Wednesday, August 28. We began a discussion of limits and continuity for functions of several variables. As a reminder, we first discussed the concepts of limit and continuity, for a function of one variable. Just as $\lim_{x\to a} f(x) = L$ intuitively mean that the values of $f(x)$ approach L as x approaches x, it should be the case that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that the values of $f(x,y)$ approach the real number L, as (x,y) approaches (a,b). We noted that while limits for functions of one variable involve a approaching x from either the right or left, for limits with functions of two variables, there are infinitely many ways (x, y) can approach (a, b). On the other hand, using the notion of distance, the definition of a limit still takes a very similar form as in one variable: Namely, we say $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$, such that $|(x, y) - (a, b)| < \delta$ implies $|x - L| < \epsilon$. We then defined $f(x, y)$ to be continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Before calculating some examples, we noted that the following rules for taking limits hold, and that these are the same rules that apply when taking limits of function of one variable.

Theorem 12.2.1 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = K.$$

The following limits hold.

1. Constants: $\lim_{(x,y)\to(x_0,y_0)} b = b$
2. Identity $\lim_{(x,y)\to(x_0,y_0)} x = x_0$; $\lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. Sums/Differences: $\lim_{(x,y)\to(x_0,y_0)} \big(f(x, y) \pm g(x, y)\big) = L \pm K$
4. Scalar Multiples: $\lim_{(x,y)\to(x_0,y_0)} b \cdot f(x, y) = bL$
5. Products: $\lim_{(x,y)\to(x_0,y_0)} f(x, y) \cdot g(x, y) = LK$
6. Quotients: $\lim_{(x,y)\to(x_0,y_0)} f(x, y)/g(x, y) = L/K$, $(K \neq 0)$
7. Powers: $\lim_{(x,y)\to(x_0,y_0)} f(x, y)^n = L^n$

Thus, for example,

$$\begin{aligned}
\lim_{(x,y)\to(1,2)} \frac{3x^2y + 4xy}{x + y} &= \frac{\lim_{(x,y)\to(1,2)} 3x^2 + 4xy}{\lim_{(x,y)\to(1,2)} x + y} \\
&= \frac{3\lim_{(x,y)\to(1,2)} x^2 + 4\lim_{(x,y)\to(1,2)} xy}{\lim_{(x,y)\to(1,2)} x + \lim_{)x,y)\to(1,2)} y} \\
&= \frac{3(\lim_{x\to1} x)^2 + 4(\lim_{x\to1} x)(\lim_{y\to2} y)}{\lim_{x\to1} x + \lim_{y\to2} y} \\
&= \frac{3 \cdot 1^2 + 4 \cdot 1 \cdot 2}{1 + 3} \\
&= \frac{11}{3}.
\end{aligned}$$

We also noted that when applying the rules above to $\lim_{(x,y)\to(a,b)} f(x, y)$ it is crucial to use limits, and not merely substitute (a, b) for (x, y). For example, let

$$f(x) = \begin{cases} x, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1, \end{cases}$$

so that $\lim_{x\to1} f(x) = 1 \neq f(1)$. Thus, $\lim_{(x,y)\to(1,2)} f(x) \cdot (x^2 + 3xy) = 1 \cdot (1^2 + 3 \cdot 1 \cdot 2) = 7$.

We then noted that the same rule above for limits apply for continuity, e.g., sum, products etc of functions continuous at (a, b) are continuous at (a, b). We finished class by calculating some examples and proving properties 2 and 3 above.

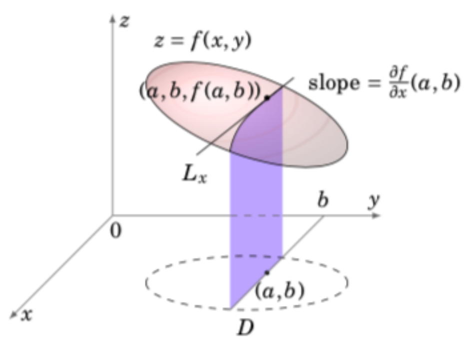
Friday, August 30. We reviewed the definitions of limit and continuity for scalar functions of the form $f(x, y)$. After doing an example where the function was continuous so that the limit was obtained by substitution, we considered the following limit: $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$. We noted that the limit was zero as (x, y) approached (0,0) along any line $y = kx$, for $k \in \mathbb{R}^2$, but that the limit is not zero as (x, y) approached (0,0) along the curve (y^2, y). This shows that for $\lim_{(x,y)\to(a,b)} f(x, y)$ to exist, it is not just a matter of the limit existing as (x, y) approached (a, b) from all possible directions.

We then discussed limits and continuity for functions of the most general type, $f : \mathbb{R}^n \to \mathbb{R}^m$, noting that the concept of distance renders the corresponding definitions almost identical to the case of $f(x, y)$ previously discussed. We then recorded the fact that if $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$, then writing $\underline{x}$ for $(x_1, \dots, x_n)$, and taking $P \in \mathbb{R}^n$, $Q = (Q_1, \dots, Q_m) \in \mathbb{R}^m$, $\lim_{\underline{x}\to P} f(\underline{x}) = Q$ if and only if $\lim_{\underline{x}\to P} f_j(\underline{x}) = Q_j$, for all $1 \le j \le m$. We ended class by giving a proof in the special case $n = m = 2$ that if the $\lim_{(x,y)\to P} f(x, y) = (Q_1, Q_2)$, then $\lim_{(x,y)\to P} f_1(x, y) = Q_1$.

Wednesday, September 4. We began our discussion of partial derivatives. We noted that given a function $f(x, y)$, if we start at the point (a, b), then there are infinitely many directions moving away from (a, b) for which we could seek the rate of $f(x, y)$. Our analysis began with the rate of change of $f(x, y)$ at (a, b) in a direction parallel to the x-axis. We noted that this can be analyzed by intersecting the graph of $z = f(x, y)$ with the plane $y = b$. We observed that doing so reduces the problem to calculating the derivative of the function $c(x) := f(x, b)$ at $x = a$. Thus,

$$c'(a) = \lim_{h\to 0} \frac{c(a + h) - c(a)}{h} = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

If this limit exists, this is the *partial derivative* of $f(x, y)$ with respect to x at (a, b), which we denote by $\frac{\partial f}{\partial x}(a, b)$ or $f_x(a, b)$. Geometrically, we can think of $\frac{\partial f}{\partial x}(a, b)$ as the slope of the line passing through the point $(a, b, f(a, b))$ tangent to the curve $z = f(x, b)$ lying on the graph of $z = f(x, y)$.



(a) Tangent line L_x in the plane $y = b$

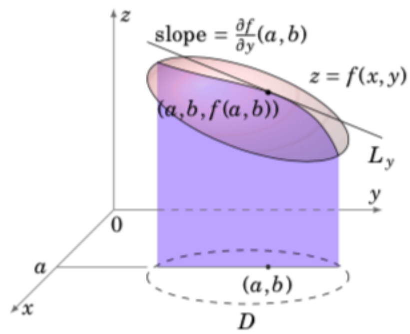
We then used the limit definition to calculate $f_x(1, 1)$ for the function $f(x, y) = 2x + 3y$ and $g_x(2, -1)$ for the function $g(x, y) = x^3y + 3xy^2$. The resulting values were 2 and -9. We then repeated the limit calculations to find the general values of $f_x(x, y)$ and $g_x(x, y)$, and observed that $f_x(x, y) = 2$ and $g_x(x, y) = 3x^2y + 3y^2$. This suggests that in general, when the relevant limits exist, the partial derivative of an arbitrary $f(x, y)$ with respect to x is obtained by differentiating the expression for $f(x, y)$ with respect to x, treating y as a constant. We then did this direct calculation for $h(x, y) = 6x^2y^3e^{x^2+2y^2} + 5\cos(xy)$, obtaining

$$\frac{\partial h}{\partial x} = 12xy^3e^{x^2+2y^2} + 6x^2y^3e^{x^2+2y^2}(2x) - 5\sin(xy) \cdot y.$$

We then noted that the discussion above applies equally well to the variable y, so that one defines the partial derivative of $f(x, y)$ with respect to y at (a, b) as

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h},$$

provided the limits exists. This then represents the slope of the line tangent tangent to the graph $z = f(x, y)$ at (a, b) along the curve obtained by intersecting the graph of $z = f(x, y)$ with the plane $x = a$.



(b) Tangent line L_y in the plane $x = a$

Accordingly, the function $\frac{\partial f}{\partial y}$ is calculated by differentiating $f(x,y)$ with respect to y, treating x as a constant. Doing so for $h(x,y)$ above yields,

$$\frac{\partial h}{\partial y} = 18x^2y^2e^{x^2+2y^2} + 6x^2y^3e^{x^2+2y^2}(4y) - 5\sin(xy)\cdot x.$$

We next noted that the definition easily carries over to functions of several variables, the general case being a function $f(x_1, x_2, \ldots, x_n)$. In this case, we defined the partial derivative of $f(x_1, \ldots, x_n)$ with respect to x_i at $(a_1, \ldots, a_n)$ to be

$$\frac{\partial f}{\partial x_i}(a_1, a_2, \ldots, a_n) = \lim_{h\to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{h},$$

if the limit exists. We then noted that to calculate partial derivatives of functions of many variables with respect to a given variable, we differentiate with respect to that variable, treating all remaining variables as constants. We then calculated some of the partial derivatives of the function

$$h(x, y, z, u, w) = x^3y^2z^5 + 3e^{xu^2w^3} + \cos(u^2 + w^2 + x^2),$$

obtaining:

(i) $\frac{\partial h}{\partial x} = 3x^2y^2z^5 + 3e^{xu^2w^3}\cdot(u^2w^3) + 0.$

(ii) $\frac{\partial h}{\partial w} = 0 + 3e^{xu^2w^3}\cdot(3xu^2w^2) - \sin(u^2 + w^2 + z^2)\cdot 2w.$

(iii) $\frac{\partial h}{\partial z} = 5x^3y^2z^4 + 0 - \sin(u^2 + w^2 + z^2)\cdot 2z.$

We ended class by noting that the following rules for partial differentiation (and their more general counter parts) follow from the corresponding familiar rules from Calculus 1.

For functions $f(x,y)$, $g(x,y)$, $h(t)$:

(i) $(f + g)_x = f_x + g_x$
(ii) $(fg)_x = f_xg + fg_x$
(iii) $(h(f(x,y))_x = h'(f(x,y))\cdot f_x(x,y)$,

and similarly for partial with respect to y.

Friday, September 6. We began class by reviewing the definition of the partial derivative of a (scalar) function of many variables. We then used the fact that for the function $f(x,y)$, if $f_x(a,b)$ exists, $f_i(a,b)$ as the slope of the line passing through the point $(a,b,f(a,b))$ tangent to the curve $z = f(x,b)$ lying on the graph of $z = f(x,y)$ in the plane $y = b$. From this, we derived the parametric equation for this line: $L_x(t) = (a,b,f(a,b)+t\cdot(1,0,f_x(a,b))$. Similarly, we noted that if $f_y(a,b)$ exists, $L_y(t) = (a,b,f(a,b))+t\cdot(0,1,f_y(a,b))$ gives a tangent line in the y direction.

We then noted that having good tangent lines in two directions leads to the expectation of having a tangent plane at the point $(a,b,f(a,b))$. After recalling the fact that a plane is determined by a point and a normal vector, we arrived at the equation for what *should be* the relevant tangent plane:

$$z = f_x(a,b)(x - a) + f_y(a,b)(y - b) + f(a,b).$$

We noted that geometrically, the cone $z = \sqrt{x^2+y^2}$ clearly does not have a tangent plane at (0,0,0), and this was confirmed by verifying that neither partial derivative exists at (0,0). We then pointed out that the situation is more subtle than this, by considering the function $f(x,y)||x|-|y||-|x|-|y|$. In this case, $f_x(0,0)$ and $f_y(0,0)$ exist, and equal 0, so that $z=0$ is the expected tangent plane. We then noted that there is no tangent plane in this case, because the curve $z = f(x,2x)$ on the graph of $f(x,y)$ does not have a a tangent line at (0,0,0). **Thus, we emphasized that the existence of partial derivatives does not guarantee the existence of a tangent plane.** We then noted that this difficulty is taken care of in the same way as in Calculus I by using the concept of differentiability. We ended class by stating the following:

Definition. Given $f(x,y)$ and (a,b) in the domain of $f(x,y)$. Suppose that $f_x(a,b)$ and $f_y(a,b)$ exist and set $L(x,y) := f_x(a,b)(x-a)+f_y(a,b)(y-b)+f(a,b)$. Then $f(x,y)$ is *differentiable* at (a,b) if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-L(x,y)}{||(x,y)-(a,b)||} = 0.$$

We observed that the numerator and denominator in the limit above both go to zero, so that if the quotient goes to zero, $f(x,y)-L(x,y)$ is going to zero significantly faster than the denominator. Thus, we concluded that if $f(x,y)$ is differentiable at (a,b), then:

(i) The function $L(x,y)$ is a good linear approximation to $f(x,y)$ for points (x,y) sufficiently close to (a,b).

(ii) The proposed tangent plane $z = L(x,y)$ is, in fact, tangent to the graph of $f(x,y)$ at (a,b).

Monday, September 9. We began class by reviewing the definition given in the previous lecture regarding the differentiability of $f(x,y)$ at (a,b), noting $L(x,y)$, as previously defined, is a good linear approximation to $f(x,y)$ at (a,b). Geometrically, this means that there is a well defined tangent plane to the graph of $z = f(x,y)$ at $(a,b,f(a,b))$. In other words, does the proposed tangent plane

$$z = L(x,y) := f_x(a,b)(x-a)+f_y(a,b)(y-b)+f(a,b)$$

does a good job of approximating $f(x,y)$ in the vicinity of (a,b).

We then noted that if $f(x,y)$ is differentiable at (a,b) and Δx and Δy are small, then

$$\begin{aligned} f(a+\Delta x, b+\Delta y) &\cong L(a+\Delta x, b+\Delta y)\\ &= f_x(x,b)(a+\Delta x-a)+f_y(a,b)(b+\Delta y-b)+f(a,b)\\ &= f_x(a,b)\Delta x+f_y(a,b)\Delta y. \end{aligned}$$

We then illustrated this with the following example.

Example. Use the linear approximation to estimate $f(1.01, 2.02)$, for $f(x,y) = x^2+2xy$. We assume that $f(x,y)$ is differentiable at $(1,2)$. We calculated $f_x(x,y) = 2x+2y$, so $f_x(1,2) = 6$; $f_y(x,y) = 2x$, so $f_y(1,2) = 2$. We also have $\Delta z = .01$ and $\Delta y = .02$. Thus,

$$f(1.02, 2.02) \cong f_x(1,2)(.01)+f_y(12)(.02)+f(1,2) = 6(.01)+2(.02)+5 = 5.1.$$

Note that $f(1.01, 2.02) = (1.01)^2+2(1.01)(2.02) = 5.1005$, so the approximation 5.1 is a good one.

We then reviewed the situation for the one variable case, as a means to see its connection with the two variable definition of differentiability. Suppose $f'(a)$ exists and we set $L(x) = h'(a)(x-a)+f(b)$, the tangent line to the graph of $y = f(x)$ at $x = a$. Since$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, we showed

$$\begin{aligned} \lim_{x\to a}\frac{f(x)-L(x)}{x-a} &= \lim_{x\to a}\frac{f(x)-\{f'(a)(x-a)+f(a)\}}{x-a}\\ &= \lim_{x\to a}\frac{f(x)-f(x)}{x-a}-f'(a)\\ &= f'(a)-f('a) = 0. \end{aligned}$$

We then easily showed that:

(a) $f(x,y) = x$ and $g(x,y) = y$ are differentiable at any (a,b) and recorded the facts

(b) If $f(x,y)$ and $g(x,y)$ are differentiable at (a,b), then so are:

(i) $f(x,y)+g(x,y)$, $f(x,y)g(x,y)$, $\frac{f(x,y)}{g(x,y)}$, if $g(a,b)\neq 0$.
(ii) $h(f(x,y))$, if $h(t)$ is differentiable at $f(a,b)$.

We followed this with a direct limit calculation showing that $f(x,y)=xy$ is differentiable at any point (a,b). A key point was the substitution $x=r\cos(\theta)+a$ and $y=r\sin(\theta)+b$.

We then stated the following **very important** theorem:

Theorem. (Differentiability Criterion) Given $f(x,y)$ and (a,b) in the domain of $f(x,y)$ and suppose $f(x,y)$ has the property that $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous in an open disk about (a,b). Then $f(x,y)$ is differentiable at all points in that disk, including the point (a,b)

To illustrate the connection between the conditions in the theorem, we considered the following example

Example. Let $f(x,y)=\begin{cases}\frac{2xy}{\sqrt{x^2+y^2}}, \text{ if } (x,y)\neq(0,0)\\ 0, \text{ if } (x,y)=(0,0)\end{cases}$. We showed:

(a) $f(x,y)$ is continuous at (0,0).
(b) $f_x(0,0)=0=f_y(0,0)$.
(c) $L(x,y)=0$.
(d) $f(x,y)$ is not differentiable at $(0,0)$.
(e) The partial derivative $f_x(x,y)$ is not continuous at (0,0).

Wednesday, September 11. We began class by recalling the criterion for differentiability presented in the previous lecture and how it applies to the the function $f(x,y)=\begin{cases}\frac{x^3y-xy^3}{x^2+y^2}, \text{ if } (x,y)\neq(0,0)\\ 0, \text{ if } (x,y)=(0,0)\end{cases}$. We showed that $f(x,y)$ is continuous at (0,0), and we calculated $f_x(x,y)$ for all values of (x,y) and noted that $f_x(x,y)$ is continuous throughout $\mathbb{R}^2$. The same applies to $f_y(x,y)$, so $f(x,y)$ is differentiable at throughout $\mathbb{R}^2$.

We then recorded the fact that if $f(x,y)$ is differentiable at (a,b), then $f(x,y)$ is continuous at (a,b). We gave the following sketch of the reason why: We noted that we must show $\lim_{(x,y)\to(0,0} f(x,y)=f(a,b)$, or that $\lim_{(x,y)\to(0,0)} f(x,y)-f(a,b)=0$. Then, if we set $L(x,y)=f_x(a,b)(x-a)+f_y(a,b)(y-b)+f(a,b)$, we have

$$\begin{aligned}\lim_{(x,y)\to(a,b)} f(x,y)-f(a,b) &= \lim_{(x,y)\to(a,b)} f(x,y)-L(x,y)+\{f_x(a,b)(x-a)+f_y(a,b)(y-b)\}\\ &= \lim_{(x,y)\to(a,b)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-a)^2+(y-b)^2}}\cdot\sqrt{(x-a)^2+(y-b)^2}+\{f_x(a,b)(x-a)+f_y(a,b)(y-b)\}.\end{aligned}$$

The last limit is of the form $\lim_{(x,y)\to(a,b)}\{A(x,y)B(x,y)+C(x,y)\}$ where the limit of each of the three terms is zero. Thus, the required limit is zero, so $f(x,y)$ is continuous at (a,b).

We then discussed how to extend the notion of differentiability to more general functions, including scalar and vector functions of several variables. The ultimate definition was as follows:

General Definition of Differentiablity. Suppose $F(x_1,\dots,x_n)=(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $P=(a_1,\dots,a_n)\in\mathbb{R}^n$, and write $\underline{x}$ for $(x_1,\dots,x_n)$. Then $F(\underline{x})$ is differentiable at P if:

(i) Each partial derivative $\frac{\partial f_i}{\partial x_j}(P)$ exists and
(ii) $\lim_{\underline{x}\to P}\frac{||F(\underline{x})-L(\underline{x})||}{||\underline{x}-P||}=0$, where $L(\underline{x})=(L_1(\underline{x}),\dots,L_n(\underline{x}))$ and each

$$L_i(\underline{x})=\frac{\partial f_i}{\partial x_1}(P)(x_1-a_1)+\cdots+\frac{\partial f_i}{\partial x_n}(P)(x_n-a_n)+f_i(P).$$

When then addressed the following question: If a function can be differentiable, is there a derivative? The answer was yes, the derivative at a point is just a matrix of partial derivatives. Thus, if $f(x,y)$ is differentiable at (a,b), the derivative $Ff(a,b)$ is the 1×2 matrix $(f_x(a,b)\ \ f_y(a,b))$. In the general case, maintaining the notation above, if $F(\underline{x})$ is differentiable at P,then its derivative is the $m\times n$ matrix

$$DF(P) = (\frac{\partial f_i}{\partial x_j}(P)),\ \text{with } 1\le i\le m \ \text{ and } \ 1\le j\le n.$$

So, for example, if $F(x,y,z)=(f(x,y,z),g(x,y,z))$, then $DF(a,b,c)=\begin{pmatrix} f_x(a,b,c) & f_y(a,b,c) & f_z(a,b,c)\\ g_x(a,b,c) & g_y(a,b,c) & g_z(a,b,c)\end{pmatrix}$. The point is that the derivative $DF(P)$ encodes exactly the information needed to write the best linear approximation of the function $F(\underline{x})$ near the point P.

We then began a discussion of optimization of functions of several variables, by noting that in the case of $f(x,y)$, a relative maximum or minimum value should occur where the tangent plane is parallel to the xy-plane, i.e., has the form $z=c$, for some $c\in\mathbb{R}$. This lead to the definition:

Definition. A point (a,b) in the domain of $f(x,y)$ is a *critical point* if either: (i) $f_x(a,b)=0=f_y(a,b)$ or (ii) One of $f_x(a,b)$ or $f_y(a,b)$ is not defined. We ended class by finding the critical point for the function $f(x,y)=3x^2+6xy+4y$.

Friday, September 13. We began class by reviewing the definition of critical point for a function $f(x,y)$. These were points (a,b) in the domain of $f(x,y)$ for which $f_x(a,b)=0=f_y(a,b)$ or for which one of f_x or f_y are undefined at (a,b). We then computed critical points the following examples.

(i) $f(x,y)=x^2+2xy-4y^2+4x-6y+4$, which has critical point $(-1,-1)$.
(ii) $f(x,y)=\sqrt{xy}$,whose critical points are the x and y axes.
(iii) $f(x,y)=(x^y-y^2)e^{-\frac{1}{2}(x^2+y^2)}$, which has critical points $(0,\,0)$, $(\pm\sqrt{2},0),(0,\pm\sqrt{2})$.

We then defined, what it means for a point to be a relative maximum or relative minimum of $f(x,y)$, or a saddle point on the graph of $f(x,y)$.

If there is an open disk D containing P such that $f(x_0,y_0)\ge f(x,y)$ for all points (x,y) that are in both D and S, then f has a **relative maximum** at P.

If there is an open disk D containing P such that $f(x_0,y_0)\le f(x,y)$ for all points (x,y) that are in both D and S, then f has a **relative minimum** at P.

Let $P=(x_0,y_0)$ be in the domain of f where $f_x=0$ and $f_y=0$ at P. We say P is a **saddle point** of f if, for every open disk D containing P, there are points (x_1,y_1) and (x_2,y_2) in D such that $f(x_0,y_0)>f(x_1,y_1)$ and $f(x_0,y_0)<f(x_2,y_2)$.

Before moving on, we defined the second order partial derivatives of $f(x,y)$:

1. The **second partial derivative of f with respect to x then x** is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

2. The **second partial derivative of f with respect to x then y** is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Similar definitions hold for $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

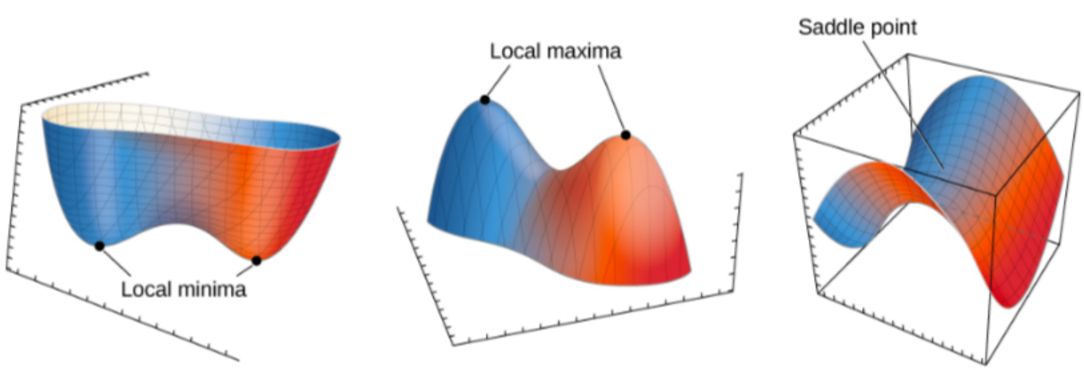
We noted that that in many cases, $f_{xy} = f_{yx}$, but this need not always hold. We stated, but did not discuss at length, that $f_{xy}(a,b) = f_{yx}(a,b)$ provided both mixed partial derivatives are continuous in an open disk about (a,b). This condition is implicit in the second derivative test below.

Theorem 12.8.2 Second Derivative Test

Let R be an open set on which a function $z = f(x,y)$ and all its first and second partial derivatives are defined, let $P = (x_0, y_0)$ be a critical point of f in R, and let

$$D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0).$$

1. If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, then f has a relative minimum at P.
2. If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, then f has a relative maximum at P.
3. If $D < 0$, then f has a saddle point at P.
4. If $D = 0$, the test is inconclusive.



Using the second derivative test, we then classified the critical points for $f(x,y) = (x^2 - y^2)e^{-\frac{1}{2}(x^2+y^2)}$ and found that $(0,0)$ is a saddle point, $f(\pm\sqrt{2},0)$ are relative minima and $f(0,\pm\sqrt{2})$ are relative maxima.

Monday, September 16. After reviewing the process for finding relative extreme values for a function $f(x,y)$ using the second derivative test, as stated in the previous lecture, we noted that one requires that the first and second order partial derivatives of $f(x,y)$ must be continuous in order for the test to be applicable. We then worked the following examples.

Examples. (i) Find and classify the critical points for $f(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$.

(ii) Find the dimensions of the rectangular box with minimum surface area, subject to the constraint that its volume is 100cm^3.

We then illustrated the second derivative test by analyzing the the values of $f(x, y) = ax^2 + 2bxy + xy^2$ near the origin by completing a square, and considering when $ac - b^2$ is greater than, less than, or equal to zero, arriving at conclusions that mirrored the second derivative test, since, using the notation from the second derivative test above, $AC - B^2 = 4(ac - b^2)$, $A = 2a$, $B = 2b$ and $C = 2c$.

We then began a discussion concerning absolute maxima and absolute minima for a function of two variables. We recalled, that for a function $f(x)$ of one variable, in order to guarantee that $f(x)$ has absolute extreme values, one must assume that $f(x)$ is continuous on a closed interval $[c, d]$. To find these values, one must find critical points on the interior of the interval $[c, d]$ and evaluate $f(x)$ at each of these points, and then one must calculate$f(c)$ and $f(d)$. The largest of these values is the absolute maximum of $f(x)$ on $[c, d]$ and smallest is the absolute minimum of $f(x)$ on $[c, d]$. We then explained that one needs similar condition for functions of two variables. Namely, the function in question must be continuous and the domain in $\mathbb{R}^2$ must be *closed* and *bounded*.

Wednesday, September 18. We began by defining the notions of *absolute maximum* and *absolute minimum*: Given a subset $D \subseteq \mathbb{R}^2$, $f(x, y)$ has an *absolute maximum* (respectively, *absolute minimum*) at (a, b) if $f(x, y) \leq f(a, b)$ (respectively, $f(a, b) \leq f(x, y)$), for all $(x, y) \in X$. In this case, $f(a, b)$ is the absolute maximum (respectively, absolute minimum) value of $f(x, y)$ on X. We then gave the following definition:

Definition. Suppose that X is a subset of $\mathbb{R}^2$.

(i) X is *bounded* if there exists a closed disk $D \subseteq \mathbb{R}^2$ with $X \subseteq D$.
(ii) If X is bounded, then X is *closed* if it contained all of its boundary points.

Thus, for example,

(i) The open disk $X_1 = \{0 < x^2 + y^2 < 1\}$ is bounded, but not closed.
(ii) The infinite vertical strip $X_2 = \{(x, y) \mid 0 \leq x \leq 2\}$ is closed, but not bounded.
(iii) The rectangle $X_3 = \{(x, y) \mid -1 \leq x \leq 2,\ -1 \leq y \leq 1\}$ is both closed and bounded.

The important theorem concerning absolute extreme values is the following:

Theorem. *Let $X \subseteq \mathbb{R}^2$ be bounded and closed and $f(x, y)$ a continuous function defined on X. Then $f(x, y)$ has both an absolute maximum and absolute minimum value on X.*

We then explained, that the process for finding the absolute extreme values of $f(x, y)$ on X is similar to the one for functions of one variable: First find the critical points on the interior of X, and then find the absolute extreme values of $f(x, y)$ along the boundary of X. The largest and smallest of these values give the required absolute maximum and absolute minimum values of $f(x, y)$ on X. We explained that often, finding the absolute extreme values of $f(x, y)$ along the boundary of X reduces to the one variable case. We then illustrated the theorem for $f(x, y) = x^2 + y^2$ and D the unit closed disk in $\mathbb{R}^2$ and for $f(x, y) = x^2 + 2xy$ and D the rectangle in $\mathbb{R}^2$ with vertices (2, -1), (2, 2), (-1, 2), (-1, -1).

We then began a discussion of whysecond derivative test works, augmenting the discussion we had in the previous lecture. Recalling Taylor's Theorem from Calculus I, we noted that, given $f(x)$ and $x = a$, under suitable differentiability conditions, a good *quadratic approximation* of $f(x)$ near $x = a$ is given by

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

We noted that if $f'(a) = 0$, i.e., $f(x)$ has a critical point at $x = a$, then the parabola $y = f(a) + \frac{1}{2}f''(a)(x-a)^2$ approximates $f(x)$ near $x = 0$. This shows that if $f''(a) > 0$, then the graph of $f(x)$ at $x = a$ is concave up, while the graph is concave down if $f''(a) < 0$.

We then stated that a multivariable version of Taylor's Theorem guarantees that, under suitable differentiability conditions, a good quadratic approximation of $f(x, y)$ for (x, y) sufficiently close to (a, b) is given by

$$(*) \quad f(x, y) \approx f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) + \frac{1}{2}\{f_{xx}(a, b)(x-a)^2 + 2f_{xy}(a, b)(x-a)(y-b) + f_{yy}(a, b)(y-b)^2\}.$$

If we let $Q(x,y)$ denote the right hand side of the expression above, i.e., $Q(x,y)$ is the good quadratic approximation of $f(x,y)$ near (a,b), Taylor's theorem states that if all first and second partials exists and are continuous in an open disk containing (a,b), then

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-Q(x,y)}{|(x,y)-(a,b)|^2} = 0,$$

which is stronger than the condition required for differentiability at (a,b). If we take h_1, h_2 sufficiently small, then (*) yields,

$$(**) \quad f(a+h_1, b+h_2) \approx f(a,b) + f_x(a,b)h_2 + f_y(a,b)h_2 + \frac{1}{2}\tilde{Q}(h_1,h_2),$$

where $\tilde{Q}(h_1,h_2) = f_{xx}(a,b)h_1^2 + 2f_{xy}(a,b)h_1h_2 + f_{yy}(a,b)h_2^2$. We then noted that our algebraic discussion from the previous lectures shows that conditions $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ hold, then $\tilde{Q}(h_1,h_2) > 0$ for all h_1, h_2 sufficiently small, showing that if $f_x(a,b) = 0 = f_y(a,b)$, then $f(a,b) < f(a+h_1, b+h_2)$ for all h_1, h_2 sufficiently small, implying that $f(x,y)$ has a local minimum at (a,b). Similarly, the other conditions in the second derivative test imply $\tilde{Q}(h_1,h_2) < 0$, for all sufficiently small h_1, h_2 or $\tilde{Q}(h_1,h_2)$ takes both positive and negative values, yielding a relative maximum or saddle point at (a,b).

We then discussed finding extreme values for functions $f(x,y,z)$ of three variables. We noted that a point $P = (a,b,c)$ is a critical point if $f(x,y,z)$ if it is either a solution to the system of equations

$$\begin{aligned} f_x(x,y,z) &= 0 \\ f_y(x,y,z) &= 0 \\ f_z(x,y,z) &= 0, \end{aligned}$$

or one of the first order partials is undefined at P. We then defined $D(P) = f_{xx}(P)f_{yy}(P) - f_{xy}(P)^2$ and

$$H(P) = \begin{vmatrix} f_{xx}(P) & f_{xy}(P) & f_{xz}(P) \\ f_{yx}(P) & f_{yy}(P) & f_{yz}(P) \\ f_{zx}(P) & f_{zy}(P) & f_{zz}(P) \end{vmatrix}.$$

$H(P)$ is called the *Hessian* of $f(x,y,z)$ at P.

Second Derivative Test. Suppose $P = (a,b,c)$ critical point that is a solution to the system of equations above and all second order partial derivatives of $f(x,y,z)$ are continuous in a disk about P. Then:

(i) If $f_{xx}(P), D(P), H(P)$ are all greater than zero, $f(x,y,z)$ has a relative minimum at P.
(ii) If $f_{xx}(P) < 0, D(P) > 0, H(P) < 0$, then $f(x,y,z)$ has a relative maximum at P.
(iii) If $H(P) \neq 0$, and neither (i) nor (ii) holds, then $f(x,y,z)$ has a saddlle point at P.
(iv) If $H(P) = 0$, the test is inconclusive.

Friday, September 20. We we began a discussion of the chain rule for multivariable functions, starting with the simplest version where $f(x,y) = f(x(t), y(t))$, leading to the formula

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

We illustrated this formula by calculating $\frac{\mathrm{d}f}{\mathrm{d}t}$, for $f(x,y) = 3x^2y^3$, with $x = 2t+1$ and $y = 2t^2$. This was followed by a sketch of the proof of the formula above, the key point being an application of the mean value theorem to $f(x,y)$, with first the x value fixed, and then the y value fixed. We then noted the following more general form: If $f = f(x_1, \ldots, x_n)$ and each $x_i = x_i(t)$, then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t} = f_{x_1}\cdot x_1'(t) + \cdots + f_{x_n}\cdot x_n'(t).$$

This ultimately lead to the most general form:

General for of the chain rule. SUpose $f = f)x_1, \ldots, x_n)$ and each $x_i = x_i(u_1, \ldots, u_m)$, then

$$\frac{\partial f}{\partial u_j} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_j} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_j}{\partial u_j},$$

for all $j = 1, 2, \ldots, m$. We then verified the chain rule in a couple of particular cases, including the cases:

(i) $f(x, y, y, z, w) = e^{x^2+yz+3w^3}$, with $x = t^2 + 5t$, $y = \cos(2t), z = 3\sin(t), w = \ln(2t+1)$, $(t > -\frac{1}{2})$.

(ii) $f(x, y, z) = 3x^2y^3z$, and $x = 3u^2 + 2vw, y = uvw^2, z = u^3 + v^3 + w^3$.

Monday, September 23. We began class by defining the directional derivative of the function $f(x, y)$ at (a, b) in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$:

$$D_{\vec{u}}f(a, b) = \lim_{h\to 0} \frac{f(a + hu_1, b + u_2h) - f(a, b)}{h},$$

noting that this is the rate of change of $f(x, y)$ at (a, b) along the line $(a, b) + t \cdot \vec{u}$, assuming the limit exists. We emphasized the importance of taking a unit vector in this definition, so that the quantity calculated only depends upon the function f and the direction of the direction vector, and not also on the magnitude of the direction vector.

We then used this definition to calculate $D_{\vec{u}}f(a, b)$ for $f(x, y) = x^2y$ in the direction of $\vec{u} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$,

$$\begin{aligned}
D_{\vec{u}}f(a, b) &= \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \\
&= \lim_{h\to 0} \frac{(a + h\frac{\sqrt{2}}{2})^2(b + h\frac{\sqrt{2}}{2}) - a^2b}{h} \\
&= \lim_{h\to 0} \frac{a^2b + abh\sqrt{2} + h^2 \cdot \frac{b}{2} + a^2h\frac{\sqrt{2}}{2} + ah^2 + \frac{\sqrt{2}}{4}h^3 - a^2b}{h} \\
&= \lim_{h\to 0} ab\sqrt{2} + h \cdot \frac{b}{2} + a^2\frac{\sqrt{2}}{2} + ah + \frac{\sqrt{2}}{4}h^2 \\
&= \sqrt{2}ab + \frac{\sqrt{2}}{2}a^2.
\end{aligned}$$

We observed that this last expression is $(2ab\vec{i} + b^2\vec{j}) \cdot \vec{u} = (\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}) \cdot \vec{u}$. We then noted that in general, one can calculate the directional derivative as

$$D_{\vec{u}}f(a, b) = (\frac{\partial f}{\partial x}(a, b)\vec{i} + \frac{\partial f}{\partial y}(a, b)\vec{j}) \cdot \vec{u}.$$

This lead to the definition of the *gradient*, ∇f, of a scalar function f:

(i) For $f(x, y)$, $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$.

(iii) For $f(x, y, z)$, $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$.

(iii) For $f(x_1, x_2, \ldots, x_n)$, $\nabla f = \frac{\partial f}{\partial x_1}\vec{e_1} + \frac{\partial f}{\partial x_2}\vec{e_2} + \cdots + \frac{\partial f}{\partial x_n}\vec{e_n}$, where the vector $\vec{e_i}$ is the vector in $\mathbb{R}^n$ all of whose coordinates are zero, except the ith coordinate, which is 1.

Using this notation, then in the cases above we have

(i) $D_{\vec{u}}f(a, b) = \nabla f(a, b) \cdot \vec{u}$.

(ii) $Duf(a, b, c) = \nabla f(a, b, c) \cdot \vec{u}$.

(iii) $D_{\vec{u}}f(a_1, a_2, \ldots, a_n) = \nabla f(a_1, a_2, \ldots, a_n) \cdot \vec{u}$.

where in each case $\vec{u}$ is an appropriate unit vector.

We then mentioned that ∇ is a *differential operator* that turns scalar value functions into vector valued function through the differentiation process. As such, one can expect ∇ to have similar properties that hold upon differentiation. Indeed, the following properties hold:

(i) $\nabla(f + g) = \nabla f + \nabla g$.

(ii) $\nabla(cf) = c\nabla f$, for the constant c.

(iii) $\nabla(fg) = f\nabla g + g\nabla f$.

(iv) For $h(t)$, $\nabla h(f) = h'(f)\nabla f$.

We ended class with two **important facts**

1. For the function $f(x, y, z)$, $D_{\vec{u}}f(a, b, c)$ achieves its greatest value when $\vec{u}$ points in the same direction as $\nabla f(a, b, c)$, and moreover, the rate of change in that direction is $||\nabla f(a, b, c)||$. Likewise, we noted that

$-\nabla f(a,b,c)$ points in the direction in which the rate of change at (a,b,c) is the least, and that rate of change is $-||\nabla f(a,b,c)||$. This followed easily from $\nabla f(a,b,c)\cdot\vec{u} = ||\nabla f(a,b,c)||\cdot||\vec{u}||\cos(\theta) = ||\nabla f(a,b,c)||\cos(\theta)$, where θ is the angle between $\nabla f(a,b,c)$ and $\vec{u}$.

2. We then considered a level surface of the form $f(x,y,z) =$ Constant, e.g., the ellipsoid $\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2} = 1$. In this case, if $P = (a,b,c)$ is a point on this surface, then $\nabla f(a,b,c)$ is normal to the surface at P. Thus, the plane tangent to the surface at P is given by the equation

$$0 = \nabla f(P)\cdot\{(x-a)\vec{i}+(y-b)\vec{j}+(z-c)\vec{k}\} = f_x(P)\cdot(x-a)+f_y(P)\cdot(y-b)+f_z(P)\cdot(z-c).$$

Wednesday, September 25. Today's class was devoted to a discussion of the following theorem.

Equality of Mixed Partials Theorem. Suppose $f(x,y)$ has continuous first and second order partials in an open disk D around the point (a,b) in its domain. Then $f_{xy}(a,b) = f_{yx}(a,b)$. In fact, for all $(x_0,y_0)\in D$, we have $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

We first illustrated this theorem with $f(x,y) = \frac{1}{x^2+y_2}$, noting that one has equality of mixed partials at all points in its domain. We then showed that $f_{xy}(0,0)$ and $f_{yx}(0,0)$, exist, but are *not* equal, for the function

$$f(x,y) = \begin{cases} x^2\arctan(\frac{y}{x}) - y^2\arctan(\frac{x}{y}), & \text{if } xy\neq 0\\ 0, & \text{if } xy = 0\end{cases}.$$

We observed that, in general, a potential problem stems from the fact that for a function $L(h,k)$, the two iterated limits $\lim_{k\to a}\lim_{h\to b} L(h,k)$ and $\lim_{h\to b}\lim_{k\to a} L(h,k)$ need not be equal - for example, $L(h,k) = \frac{h+k}{h-k}$.

We ended class by outlining what will take place later today and during the next four classes: Today, information about, and practice problems for, the first midterm exam will be distributed; Tomorrow, worksheets as usual. Friday and Monday of next week, working on practice problems for the midterm; Turn in Math Diaries on Monday; Tuesday, lab session, no quiz and midterm in the evening.

Friday, September 27. Today the class worked on practice problems for Exam I.

Monday, September 30. Today the class worked on practice problems for Exam I.

Wednesday, October 2. We began class with comments concerning the midterm exam. In particular, we worked problem 4 on the exam. We also gave the following bonus problem worth 5 points to be turned in at the start of class Friday. The hint for the bonus problem is to first work the special case given in the first midterm bonus problem.

Bonus Problem. Let S be the surface that is the graph of the equation $z = f(x,y)$ and suppose that $P = (a,b,f(a,b))$ is a point on S. Let L_0 be a line in $\mathbb{R}^2$ passing through (a,b) and C denote the curve consisting of the points on S lying above L_0. Let $\vec{u} = u_1\vec{i}+u_2\vec{j}$ be a unit direction vector for L_0. Give a rigorous explanation for why

$$L(t) = (a,b,f(a,b)) + t(u_1,u_2,D_{\vec{u}}f(a,b))$$

is the parametric equation of the line tangent to C at the point P. We will assume that $f(x,y)\geq 0$ in an open disk about (a,b) (so the surface lies above the xy-plane near P) and the first order partials of $f(x,y)$ exist and are continuous in an open disk about (a,b).

We then began a discussion of finding extreme values of a function $f(x,y,z)$ subject to a constraint $g(x,y,z) = C$, for a constant C. The strategy is to use the constraint equation together with the vector equation $\nabla f(x,y,z) = \lambda\nabla g(x,y,z)$. If the resulting resulting system has solutions for some value, or values of λ, these solutions are critical points to be tested in the function $f(x,y,z)$. This is called the Method of Lagrange Multipliers, and works for functions in any number of variables. We then applied this method to the following two examples:

(i) Find the dimensions of an open top box of minimum surface area having volume 4cm^3.
(ii) Find the extreme values of $f(x,y) = \frac{x^2}{4}+y^2$ subject to $g(x,y) = x^2+y^2 = 1$.

For (ii), we found the critical points $(\pm 1, 0)$, for $\lambda = \frac{1}{4}$ and $(0, \pm 1)$, for any value of λ.

Friday, October 4. We began class with a discussion of the technique of Lagrange multipliers, and gave some indications as to why the method works, at least for functions of three variables. The point was that if S is the surface defined by the constraint equation $g(x, y, z) = c$, for c a constant, and $P \in S$ yields an extreme value for $f(x, y, z)$ on S, then $f(x(t), y(t), z(t))$ has an extreme value at P for any curve $C(t) = (x(t), y(t), z(t))$ on S passing through P. Using the chain rule to differentiate $f(C(t))$ shows that $\nabla f(P)$ is orthogonal to $C'(t)$, the tangent to the curve at P. Doing this for two curves with distinct tangents at P shows $\nabla f(P)$ is normal to the tangent plane to S at P and thus a multiple of $\nabla g(P)$, which explains why we use the equation $\nabla f = \lambda \nabla g$ in the Lagrange multiplier technique.

We then discussed how the technique works when one wishes to find the extreme values for $f(x, y, z)$ subject to two constraints, $g_1(x, y, z) = c_1$ and $g_2(x, y, z) = c_1$. In this case, if $f(P)$ is an extreme value for $f(x, y, z)$ subject to the given constraints, then there exist $\lambda_1, \lambda_2 \in \mathbb{R}^2$ such that P is a solution to the system of equations

$$
\begin{aligned}
\nabla f(x, y, z) &= \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z) \\
g_1(x, y, z) &= c_1 \\
g_2(x, y, z) &= c_2.
\end{aligned}
$$

We ended class by working the following example: Let C be the curve obtained by intersecting the cone $z^2 = x^2 + y^2$ with the plane $z = x + y + 2$. Find the points on C nearest and farthest from the origin in $\mathbb{R}^3$.

Monday, October 7. Today we started our discussion of multiple integration. We began class by reviewing the situation for definite integrals of the form $\int_a^b f(x)\ dx$. We first recalled the definition of the definite integral as a limit of partial sums taken over successively finer partitions of the interval $[a, b]$. We then demonstrated how the Fundamental Theorem of Calculus works, i.e., we sketched a proof of the crucial formula $\int_a^b f(x)\ dx = F(b) - F(a)$, for $F(x)$ satisfying $F'(x) = f(x)$. The point of this discussion being two fold: It is important to understand conceptually what a definite integral is, while on the other hand, one needs to know how to calculate a definite integral.

We then showed that one can define the definite integral of $f(x, y)$ over a closed rectangle R as a limit of partial sums of a similar type, only now one takes finer rectangular partitions of the domain R, obtaining a limit of sums of the form $\Sigma_i \Sigma_j f(c_i, d_j)\Delta x \Delta y$. This limit, if it exists, is denoted $\int\int_R f(x, y)\ dA$. We ended class by noting Fubini's theorem as a means of calculating $\int\int_A f(x, y)\ dA$:

Fubini's Theorem for rectangles. Suppose $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$. Then

$$
\int\int_D f(x, y)\ dA = \int_c^d \{\int_a^b f(x, y)\ dx\}\ dy = \int_a^b \{\int_c^d f(x, y)\ dy\}\ dx,
$$

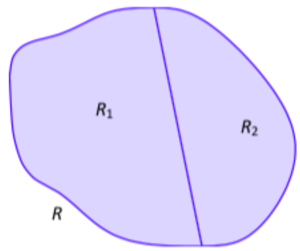
where $\int_c^d \{\int_a^b f(x, y)\ dx\}\ dy$ and $\int_a^b \{\int_c^d f(x, y)\ dy\}\ dx$ are *iterated integrals*. We discussed how to calculate iterated integrals by integrating one variable at a time, keeping the second variable fixed during the first integration process.

Wednesday, October 9. We began class by reviewing Fubini's theorem for integrating $f(x, y)$ over a rectangle and calculated $\int\int_R xye^{x^2+y^2}\ dA$ in two ways, for $R = [0, 1] \times [2.3]$. We then presented the following properties

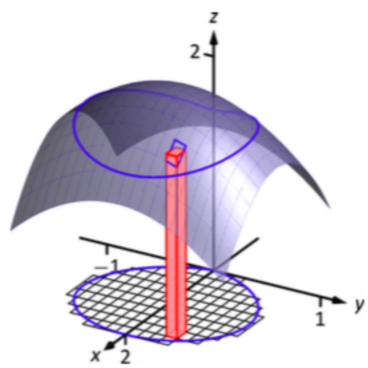
Properties. Assume $f(x, y)$, $g(x, y)$ are integrable over the rrectangular region R. Then:

(i) $\int\int_R \{f(x, y) \pm g(x, y)\}\ dA = \int\int_R f(x, y)\ dA \pm \int\int_R g(x, y)\ dA$.
(ii) $\int\int_R \lambda \cdot f(x, y)\ dA = \lambda \cdot \int\int_D f(x, y)\ dA$, for all $\lambda \in \mathbb{R}$.

(iii) $\int\int_R = \int\int_{R_1} f(x,y)\ dA + \int\int_{R_2} f(x,y)\ dA$, where $R = R_1 \cup R_2$ and either R_1, R_2 are disjoint, or only intersect along their boundaries

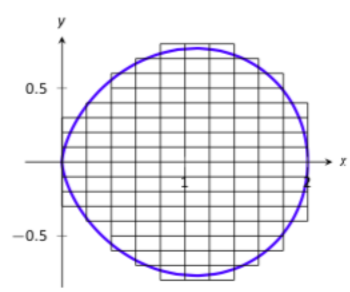


(iv) If $f(x,y) \geq 0$, for all $(x,y) \in R$, then $\int\int_D f(x,y)\ dA$ represents the volume of the region in $\mathbb{R}^3$ bounded above by the graph of $f(x,y)$ and bounded below by R. The point being that a partial sum taken over a rectangular partition of R reprsents a sum of vlumes of cubes of the type below. Taking the limit of such sums gives the indicated volume.



(b)

We then had a brief a discussion of what $\int\int_D f(x,y)\ dA$ should mean, where $R \subseteq \mathbb{R}^2$ is a possible domain of integration. Proceeding by analogy, we observed that the notation is suggestive: we should be summing (via a double sum) values of $f(x,y)$ times small increments of area. For this we described the process of covering the region R with small rectangles $\Delta x_i \times \Delta y_j$, something like this:



We selected a point (c_i, d_j) from each $\Delta x_1 \times \Delta y_j$ rectangle and formed the Riemann sum $\sum_i \sum_j f(c_i, d_j)\Delta x_i \Delta y_j$. We then defined

$$\int\int_D f(x,y)\ dA = \lim_{\Delta x, \Delta y \to 0} \sum_i \sum_j f(c_i, d_j)\Delta x_i \Delta y_j,$$

provided the limit exists. The resulting number is then *double integral of f(x,y) over the region R.*

Next we stated the following version of Fubini's theorem that enables us to calculated double integrals over more general regions:

Theorem 13.2.2 Fubini's Theorem

Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function on R.

1. If R is bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$, then

$$\iint_R f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx.$$

2. If R is bounded by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$, then

$$\iint_R f(x,y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy.$$

We ended class by calculating $\int\int_R x^2y + e^x\ dA$ in two ways, for R the triangle with vertices (0,0), (1, 0), (1,10).

Friday, October 11. We began class with the following example.

Example 1. Set up $\int\int_R e^{-x^2}\ dA$ in two ways, for R the region given by $0 \le x \le 2$ and $0 \le y \le 2x$. We have

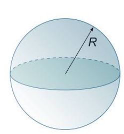
$$\begin{aligned}\int\int_R e^{-x^2}\ dA &= \int_0^2 \int_0^{2x} e^{-x^2}\ dy\ dx \\ &= \int_0^4 \int_{\frac{y}{2}} e^{-x^2}\ dx\ dy.\end{aligned}$$

We noted it is not possible to calculate the second integral. Calculating the first integral leads to

$$\begin{aligned}\int_0^2 \int_0^{2x} e^{-x^2}\ dy\ dx &= \int_0^2 \{e^{-x^2}y\}_0^{y=2x}\ dx \\ &= \int_0^2 2xe^{-x^2}\ dx \\ &= \{-e^{-x^2}\}_0^2 \\ &= -e^{-4} + 1.\end{aligned}$$

We then considered the volume of the sphere of radius R centered at the origin in $\mathbb{R}^3$.

We then turned to the question of calculating the volume of the sphere of radius R centered at the origin. (In class, we used ρ for the radius.)



We can integrate the function $f(x, y) = \sqrt{R^2 - x^2 - y^2}$ over the closed disk $D : 0 \le x^2 + y^2 \le R^2$. This will give us the volume of the top half of our sphere. We can think of D as a region of Type 2, being bounded above by the curve $y = \sqrt{R^2 - x^2}$ and bounded below by the curve $y = -\sqrt{R^2 - x^2}$, with $-R \le x \le R$.

Thus, the volume of the sphere is given by

$$2\int\int_D \sqrt{R^2-x^2-y^2}\ dA = 2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\ dy\ dx.$$

Note that the first of the two iterated integrals requires consideration of an indefinite integral of the form $\int \sqrt{a^2-y^2}\ dy$, where $a = \sqrt{R^2-x^2}$.

This can be worked out using a trig substitution like $y = a\sin(u)$, and the answer becomes

$$\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}(\frac{y}{2}).$$

We must then replace a by $\sqrt{R^2-x^2}$, take the difference of y evaluated at $\sqrt{R^2-x^2}$ and $-\sqrt{R^2-x^2}$, and then integrate with respect to x.

There is a better solution!

The idea is a two variable form of u-substitution, namely, we use polar coordinates as follows: Set $x = r\cos(\theta)$, $y = r\sin(\theta)$, $dA = rdrd\theta$. We will explain this latter equality, in a future lecture, but the point is that just like in u-substitution, we don't simply exchange dx for du, here we do not simply exchange dA for $drd\theta$, as there is a scaling factor of r involved. In terms of r and θ, D is described as: $0 \le r \le R$ and $0 \le \theta \le 2\pi$. Upon substituting, we get:

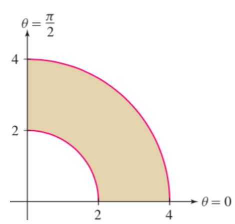
$$\begin{aligned}
2\int\int_D \sqrt{R^2-x^2-y^2}\ dA &= 2\int_0^{2\pi}\int_0^R \sqrt{R^2-(r\cos(\theta))^2-(r\sin(\theta))^2}\ r\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^R \sqrt{R^2-r^2(\cos^2(\theta)+\sin^2(\theta))}\ r\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^R r\sqrt{R^2-r^2}\ dr\ d\theta
\end{aligned}$$

Note that now the domain of integration is a rectangle in the (r,θ) plane. An easy u-substitution shows that $\int r\sqrt{R^2-r^2}\ dr = -\frac{1}{3}(R^2-r^2)^{\frac{3}{2}}$. Thus:

$$\begin{aligned}
2\int_0^{2\pi}\int_0^R r\sqrt{R^2-r^2}\ dr\ d\theta &= 2\int_0^{2\pi} -\frac{1}{3}(R^2-r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=R}\ d\theta \\
&= 2\int_0^{2\pi} 0 + \frac{R^3}{3}\ d\theta \\
&= 2\cdot\frac{R^3}{3}\theta\Big|_{\theta=0}^{\theta=2\pi} \\
&= 2\cdot\frac{2\pi R^3}{3} \\
&= \frac{4\pi}{3}R^3.
\end{aligned}$$

We then considered the following example.

Example 2. Calculate $\int\int_R x+y\ dA$, where D is the region



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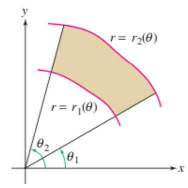
We noted that, in polar coordinates, D can be described as $0 \leq \theta \leq \frac{\pi}{2}$ and $2 \leq r \leq 4$. Thus:

$$\begin{aligned}\int\int_D x + y \; dA &= \int_0^{\frac{\pi}{2}} \int_2^4 (r\cos(\theta) + r\sin(\theta))r \; dr \; d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_2^4 r^2(cos(\theta) + \sin(\theta)) \; dr \; d\theta\end{aligned}$$

which is easily calculated.

We followed this by stating a version of Fubini's Theorem over more general polar regions. Suppose we wish to integrate the continuous function $f(x, y)$ over a region D of the following type:

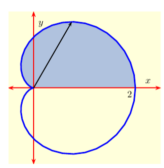


Here D is given by $r_1(\theta) \leq r \leq r_2(\theta)$ and $\theta_1 \leq \theta \leq \theta_2$, where each $r_i(\theta)$ is a function of θ. Then we have

$$\int\int_D f(x, y) \; dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta)) \; rdrd\theta.$$

We illustrated this with the following example:

Example 3. Calculate $\int\int_D y \; dA$, where D is the set of points lying above the x-axis and inside the cardioid $r = 1 + \cos(\theta)$:



Solution:

$$\begin{aligned}\int\int_D y \; dA &= \int_0^{\pi} \int_0^{1+\cos(\theta)} r\sin(\theta) rdrd\theta \\ &= \int_0^{\pi} \int_0^{1+\cos(\theta)} r^2 \sin(\theta) \; drd\theta \\ &= \int_0^{\pi} \frac{r^3}{3}\bigg|_0^{1+\cos(\theta)} \sin(\theta) \; d\theta \\ &= \frac{1}{3}\int_0^{\pi} (1 + \cos(\theta))^3 \sin(\theta) \; d\theta\end{aligned}$$

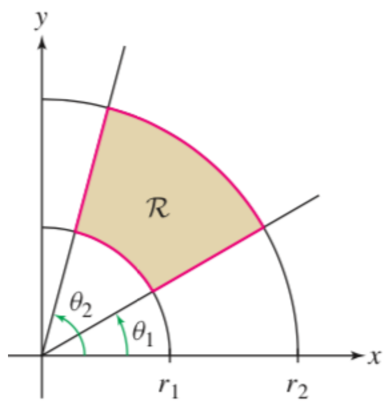
The u-substitution $u = 1 + \cos(\theta)$ yields

$$\int (1 + \cos(\theta))^3 \sin(\theta) \; d\theta = \int -u^3 \; du = -\frac{1}{4}u^4 = -\frac{1}{4}(1 + \cos(\theta))^4.$$

Thus:

$$\begin{aligned}\int\int_D y\; dA &= \frac{1}{3}\{-\frac{1}{4}(1+\cos(\theta))^4\Big|_{\theta=0}^{\theta=\pi}\}\\ &= -\frac{1}{12}\cdot\{(1+-1)^4-(1+1)^4\}\\ &= \frac{16}{12}\\ &= \frac{4}{3}.\end{aligned}$$

We ended class with a discussion explaining why dA is replaced by $rdrd\theta$ when using polar coordinates. Working in polar coordinates, it makes sense to subdivide the domain of integration D into small *polar rectangles* with $\theta_1 \leq \theta \leq \theta_2$ and $r_2 \leq r \leq r_2$, D is covered by regions $\mathcal{R}$ that look like:



When we form a Riemann sum, we must multiply a function value on $\mathcal{R}$ by the area of $\mathcal{R}$. The area of $\mathcal{R}$ is:

$$\frac{r_2^2}{2}\cdot(\theta_2-\theta_1)-\frac{r_1^2}{2}\cdot(\theta_2-\theta_1).$$

Now set $\theta_2-\theta_1=\Delta\theta$, $r=r_1$ and $r_2=r+\Delta r$. Then:

$$\begin{aligned}\text{area}(\mathcal{R}) &= \frac{(r+\Delta r)^2}{2}\cdot\Delta\theta-\frac{r^2}{2}\cdot\Delta\theta\\ &= r\ \Delta r\ \Delta\theta+\frac{(\Delta r)^2\Delta\theta}{2}.\end{aligned}$$

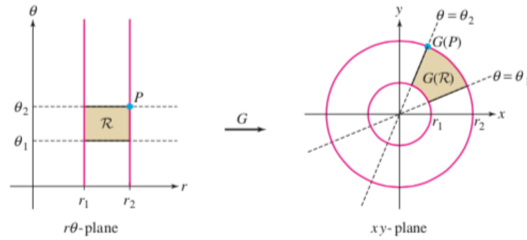
When Δr and $\Delta\theta$ are small, the term $\frac{(\Delta r)^2\Delta\theta}{2}$ is much smaller than the term $r\ \Delta r\ \Delta\theta$. Thus:

$$\text{area}(\mathcal{R}) \approx r\ \Delta r\ \Delta\theta.$$

This approximation gets better as Δr and $\Delta\theta$ tend to zero. Thus dA, measured in polar coordinates, becomes $r\ dr\ d\theta$. We can use these approximations in the Riemann sums defining the double integral, which in the limit becomes an iterated integral $\int\int f(r\cos(\theta), r\sin(\theta))\ r\ dr\ d\theta$.

Wednesday, October 16. We began our discussion of the change of variables principle for double integrals. We noted that one of the purposes of this principle is that it transforms a double integral over a domain of integration that may be difficult to integrate over into a double integral over a domain of integration that is more manageable. The example of this we have already seen is the use of polar coordinates. We can think of using polar coordinates as changing variables from x and y to r and θ. If we write $G(r,\theta) = (r\cos(\theta), r\sin(\theta))$, then we can think of G as a function that transforms verticle lines in the (r,θ)-plane to arcs on circles in the

xy-plane .

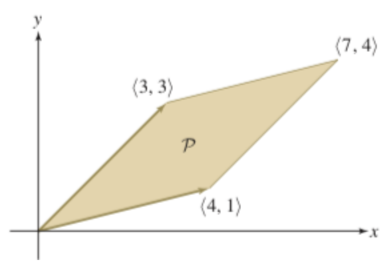


Note that $G(r, \theta)$ takes any vertical line $r = r_0$ in the r, θ-plane and wraps it infinitely many times around the circle of radius r_0 centered at the origin in the xy-plane. If $r = r_0$ and $0 \leq \theta < 2\pi$, then G applied to this vertical line seqment in the r, θ-plane is the circle of radius r_0 (no points repeated) centered at (0,0) in the xy-plane. G also takes the rectangle $\mathcal{R}$ in the uv-plane, in the diagram above, to the polar rectangle $G(\mathcal{R})$ in the xy plane.

The r in the equation $dA = r\ dr\ d\theta$ comes from the *Jacobian* of the polar transformation

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

Example 1. Consider $\int\int_{\mathcal{P}} 3x + 2y\ dA$ for $\mathcal{P}$ the region:



A close look at $\mathcal{P}$ shows that if we try to think of $\mathcal{P}$ as a region of Type 1 or Type 2, we will have to subdivide $\mathcal{P}$ into three parts.

However, we can change variables.

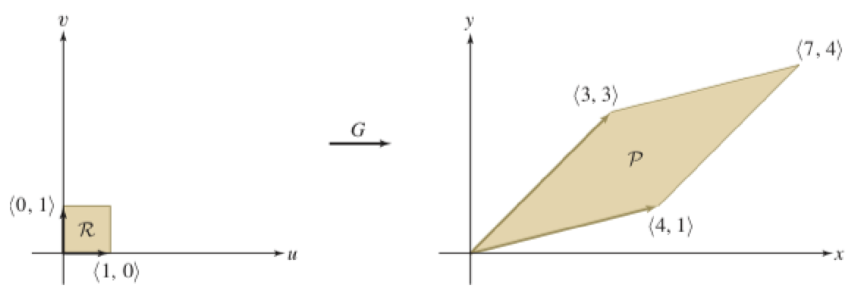
Set $x = 4u + 3v, y = u + 3v$, or equivalently, define $G(u, v) = (4u + 3v, u + 3v)$. We take the absolute value of the determinant of the 2×2 matrix of partial derivatives:

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} 4 & 3 \\ 1 & 3 \end{pmatrix} = 9 = |9|$$

and set $dA = 9\ du\ dv$. Now we substitute:

$$\begin{aligned}
\int\int_{\mathcal{P}} 3x + 2y\ dA &= \int_0^1 \int_0^1 3(4u+3v) + 2(u+3v)\ 9dudv \\
&= 9\int_0^1 \int_0^1 14u + 15v\ dudv \\
&= 9\int_0^1 (7u^2 + 15uv)\Big|_{u=0}^{u=1} dv \\
&= 9\int_0^1 7 + 15v\ dv \\
&= 9(7v + \frac{15}{2}v^2)\Big|_0^1 \\
&= 9(7 + \frac{15}{2}) \\
&= \frac{261}{2}.
\end{aligned}$$

Where does this come from? We have a transformation (function) $G : \mathbb{R}^2 \to \mathbb{R}^2$ which takes (u, v) in the uv-plane to $(4u + 3v, u + 3v)$ in the xy-plane:



Let's see how G transforms $\mathcal{R}$ to $\mathcal{P}$. Note that $\mathcal{P}$ is the parallelogram spanned by the vectors (4,1) and (3,3). $G(0, 0) = (0, 0)$, $G(0, 1) = (3, 3)$,G(1,0) = (4,1), and $G(1, 1) = (7, 4)$ showing that G takes the corners of the unit square in the uv-plane to the corners of the parallelogram $\mathcal{P}$ in the xy-plane. .

We can also verify that G takes any point on the u-axis in the uv-plane to a point on the line through (0,0) and $(4, 1)$. For example, $G(a, 0) = (4a, a)$, which lies on the line $y = \frac{1}{4}x$. Note that if $0 \le a \le 1$, then $(4a, a)$ lies on the line segment through (0,0) and (4,1). Thus, G transforms the lower edge of $\mathcal{R}$ to the line segment in $\mathcal{P}$ connecting $(0, 0)$ and $(4, 1)$.

Since $G(0, 1) = (3, 3)$ and $G(1, 1) = (7, 4)$, in a similar way one can see that G transforms each of the edges of $\mathcal{R}$ into corresponding edges of $\mathcal{P}$.

Finally, if (a, b) is in the interior of $\mathcal{R}$, then $0 < a < 1$ and $0 < b < 1$. The slope of the line through (0,0) in the xy-plane and $G(a, b)$ is $\frac{1}{4} \le \frac{u+3v}{4u+3v} \le 1$, which shows that $G(a, b)$ lies in the interior of $\mathcal{P}$.

Thus, G transforms $\mathcal{R}$ into $\mathcal{P}$.

Definition. (i) A transformation is a function $G(u, v) = (x(u, v), y(u, v))$, from the uv-plane to the xy-plane, The *Jacobian* of $G(u, v)$ is the function

$$\mathrm{Jac}(G) := \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}.$$

We will also write $\mathrm{Jac}(G) = \frac{\partial(x,y)}{\partial(u,v)}$. We will assume that our transformations satisfy the property that *all first order partial derivatives* $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ *exist and are continuous* in the domain of $G(u, v)$.

(ii) The transformation $G(u,v)$ is said to be *one-to-one* if no two points in the uv-plane go to the same point in the xy-plane under the transformation $G(u,v)$. i,e., $G(u_1,v_1) = G(u_2,v_2)$ implies $(u_1,v_1) = (u_2,v_2)$.

Here is the theorem that tells us how to change variables in a double integral.

Theorem. Let $G(u,v) = (x(u,v), y(u,v))$ be a transformation from the uv-plane to the xy-plane. Suppose D_0 is a subset of the uv-plane and write $D = G(D_0)$. Assume $G(u,v)$ is one-to-one on the interior of D_0. Then:

$$\begin{aligned}\int\int_D f(x,y)\ dA &= \int\int_{D_0} f(x(u,v), y(u,v))\ |Jac(G)|\ dudv\\ &= \int\int_{D_0} f(x(u,v), y(u,v))\ \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dudv,\end{aligned}$$

where $|\mathrm{Jac}(G)| = |\frac{\partial(x,y)}{\partial(u,v)}|$ denotes the absolute value of the Jacobian of G. The crucial point in this formula is that small portions of area dA in the xy-plane become $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ times small portions of area dA in the uv-plane.

We then looked at a special type of transformation:

Translation. Let $G(u,v) = (u+a, v+b)$. Then this is the transformation obtained by translating the origin of the uv-plane to the point (a,b) in the xy-plane. We noted that such a transformation should not change area

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1.$$

For example: If $u^2 + v^2 = 10^2$, and $G(u,v) = (u+a, v+b)$, then $u = x - a$ and $v = y - b$, so

$$(x-a)^2 + (y-b)^2 = 10^2.$$

In other words, G translates the circle (and the disk) of radius R in the uv-plane centered at (0,0) to the circle (and disk) of radius R in the xy-plane, centered at (a,b). Thus, if D_0 is the corresponding disk in the uv-plane and D is the corresponding disk in the xy-plane,

$$\int\int_D xy\ dA = \int\int_{D_0} (u+a)(v+b)\ dudv.$$

To calculate this, we need another change of variables, converting u, v to polar coordinates:

$$\begin{aligned}\int\int_D xy\ dA &= \int\int_{D_0} (u+a)(v+b)\ dudv\\ &= \int_0^{2\pi}\int_0^{10} (r\cos(\theta)+a)(r\sin(\theta)+b)\ rdrd\theta\end{aligned}$$

which can now easily be calculated.

Friday, October 18. We continued our discussion of the change of variables theorem for double integration. After recalling the translation transformation, we discussed at length linear transformations:

Linear Transformations. A transformation T from the uv-plane to the xy-plane is said to be a *linear transformation* if $T(u,v) = (au + bv, cu + dv)$, for constants $a, b, c, d \in \mathbb{R}$. Note that in this case we have

$$\mathrm{Jac}(T) = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad - bc.$$

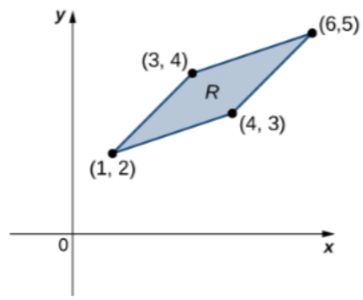
Important Properties of linear transformations.

(i) $T(P_1 + P_2) = T(P_1) + T(P_2)$, for points P_1, P_2 in the uv-plane.
(ii) $T(\lambda P) = \lambda T(P)$, for all points P in the uv-plane and $\lambda \in \mathbb{R}$.
(iii) T is one-to-one if and only if the Jacobian $ad - bc \neq 0$

Part (ii) shows that T takes lines through the origin in the uv-plane to lines through the origin in the xy-plane. In general, T transforms the unit square in the uv-plane to the parallelogram in the xy-plane spanned by the vectors (a, c) and (b, d). We noted that this can be seen using the same reasoning as in Example 1 from the previous lecture.

We then did an example showing how to construct a transformation that combines a translation and a linear transformation:

Example 1. Determine the transformation which takes the unit square in the uv-plane to the parallelogram



in the xy-plane.

Solution: We see that this parallelogram is the translation of one similar to it with lower left corner at the origin. If we move the vertex (1,2) to the origin,we get a new parallelogram with vertices (0,0), (2,2), (5,3), (3,1), moving counterclockwise along the perimeter. This new parallelogram is spanned by the vectors (3,1) and (2,2), so that by the previous lecture, $T(u, v) = (3u + 2v, u + 2v)$ takes the unit square in the uv-plane to the new parallelogram. If we add (1,2) to the new coordinates, we get

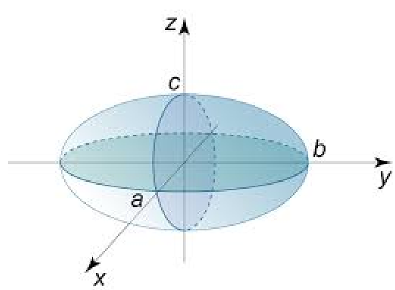
$$G(u, v) = (3u + 2v + 1, u + 2v + 2),$$

and this transformation takes the unit square in the uv-plane to the original parallelogram in the xy-plane.

Notice that G takes the vertices (0,0), (1,0), (1,1), (0,1), in the uv-plane to the vertices (1,2), (4,3), (6,5), (3,4) in the xy-plane. Moreover, $\text{Jac}(G) = \det \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = 4$.

We ended class with a detailed discussion of the transformations needed to calculate the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Example 2. Calculate the volume of the ellipsoid E: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.



Solution: If we let D be the elliptic disk $0 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, then:

$$\text{vol}(E) = 2 \int\int_D c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dA.$$

We can do this two different ways using the change of variables formula. First, we can use the transformation from the end of last lecture, $G(u, v) = (au\cos(v), bu\sin(v))$, with $|\frac{\partial(x,y)}{\partial(u,v)}| = abu$, which takes the rectangle

$0 \le u \le 1$, $0 \le v \le 2\pi$ to D. We then get:

$$\begin{aligned}
\mathrm{vol}(E) &= 2\int_0^{2\pi}\int_0^1 c\sqrt{1-\frac{(au\cos(\theta))^2}{a^2}-\frac{(bu\sin(\theta))^2}{b^2}}\ abu\ dudv \\
&= 2abc\int_0^{2\pi}\int_0^1 u\sqrt{1-u^2}\ dudv \\
&= 2\int_0^{2\pi} -\frac{1}{3}(1-u^2)^{\frac{3}{2}}\Big|_{u=0}^{u=1}\ dv \\
&= 2abc\int_0^{2\pi}\frac{1}{3}\ dv \\
&= \frac{4}{3}\pi abc
\end{aligned}$$

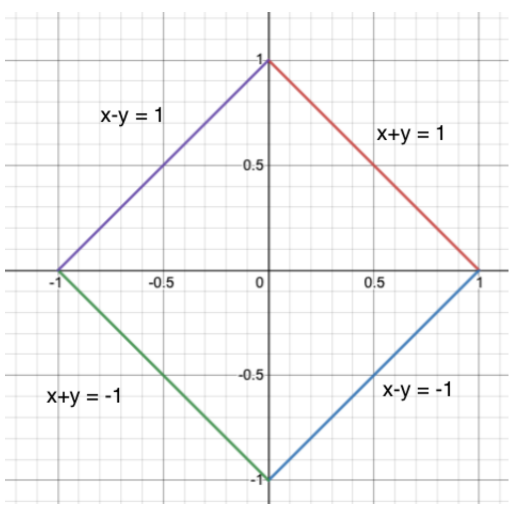
Alternately, we can use the linear transformation $T(u,v)=(au,bv)$, with $|\frac{\partial(x,y)}{\partial(u,v)}| = ab$. This transformation stretches the plane a units horizontally and b units vertically. It takes the unit disk $D' : 0 \le u^2+v^2 \le 1$ to D. Thus:

$$\begin{aligned}
\mathrm{vol}(E) &= 2\int\int_D c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\ dA. \\
&= 2c\int\int_{D'}\sqrt{1-\frac{(au)^2}{a^2}-\frac{(bv)^2}{b^2}}ab\ dudv \\
&= 2abc\int\int_{D'}\sqrt{1-u^2-v^2}\ dudv.
\end{aligned}$$

Now, we can either use polar coordinates to evaluate the last double integral, or in this case, recognize it as the volume of the top half of the unit sphere, which is $\frac{2}{3}\pi$. Thus, $\mathrm{vol}(E) = \frac{4}{3}\pi abc$.

Monday, October 21. we began with an example which shows that a change of variables can be used when the integrand has no (obvious) antiderivative.

Example 1. Calculate $\int\int_D (x+y)^2 e^{x^2-y^2}\ dA$, where D is the diamond with vertices (1,0), (-1,0), (0,1), (0,-1).



Solution: Note that as written, the integrand does not have an antiderivative with respect to either variable. If we could find a transformation $G(u,v)$ having the property that $x+y=u$ and $x-y=v$, then the integrand becomes u^2e^{uv}, which is manageable. This suggests that to find $G(u,v)$, we must solve for x and y in terms of u and v. Adding the two equations gives $2x = u+v$, so $x = \frac{u+v}{2}$. Subtracting gives $2y = u-v$, so $y = \frac{u-v}{2}$. Therefore, we take $G(u,v) = (\frac{u+v)}{2}, \frac{u-v}{2})$ is linear so the pre-image D_0 of D must at least be a parallelogram. The corners of D are (1,0), (-1,0), (0,1), (0,-1). Substituting these points into the equations for u and v gives, (1,1), (-1,-1), (1,-1), (-1,1). Thus, D_0 is the rectangle $[-1,1]\times[-1,1]$ in the uv-plane. In other words, $G(u,v)$ transforms D_0 into D.

We also have $\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}$. Thus, $|\text{Jac}(G)| = \frac{1}{2}$. Therefore,

$$\begin{aligned}
\int\int_D (x+y)^2 e^{x^2-y^2}\ dA &= \int\int_{D_0} u^2e^{uv}\ \frac{1}{2}dudv \\
&= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} u^2e^{uv}\ dvdu \\
&= \frac{1}{2}\int_{-1}^{1} ue^{uv}\Big|_{v=-1}^{v=1}\ du \\
&= \frac{1}{2}\int_{-1}^{1} u(e^u - e^{-u})\ du \\
&= 2e^{-1},
\end{aligned}$$

the last step being a standard Calculus 2 problem that can be solved using integration by parts.

We then discussed how the previous problem is related to the notion of an inverse for $G(u,v)$ We knew how to express u,v in terms of x,y, but really wanted to express x,y in terms of u,v. What this means, is that we were given a function $F(x,y) = (u(x,y), v(x,y))$ that writes u,v in terms of x,y, and we want to "unravel" $F(x,y)$ so that we can express x,y in terms of u,v, via the function $G(u,v)$. This involves realizing $G(u,v)$ as the inverse of $F(x,y)$, or equivalently, regarding $F(x,y)$ as the inverse of $G(u,v)$. This lead to the:

Definition. The transformation $F(x,y) = (u(x,y), v(x,y))$ taking points in the xy-plane to points in the uv-plane is the *inverse of G(u,v)* if $F(G(u,v) = (u,v)$ for all (u,v) in the domain of G and $G(F(x,y)) = (x,y)$ for all (x,y) in the domain of F.

It is important that both these equations hold, otherwise F is not the inverse of G. By symmetry, it follows that G is the inverse of F. While it may not always be possible to find F given G, the idea is that if we express x and y in terms of u and v, we try to solve for u and v in terms of x and y to find F. Conversely, if we are given equations expressing u and v in terms of x and y, we regard this as F, and if we can solve for x and y in terms of u and v this gives G. We checked that in the example above $F(x,y) = (x+y, x-y)$ is the inverse of $G(u,v) = (\frac{u+v}{2}, \frac{u-v}{2})$. We also noted that if D is the original domain of integration in the xy-plane, and one knows the inverse $F(x,y)$, then $F(D) = D_0$ is the domain of integration in the uv-plane.

We followed this with a discussion explaining how the Jacobian comes into the change of variables formula for double integrals. In other words, why do we use $dA = |\frac{\partial(x,y)}{\partial(u,v)}|dudv$? To see this, we started with the transformation $G(u,v) = (x(u,v), y(u,v))$ and the following two facts:

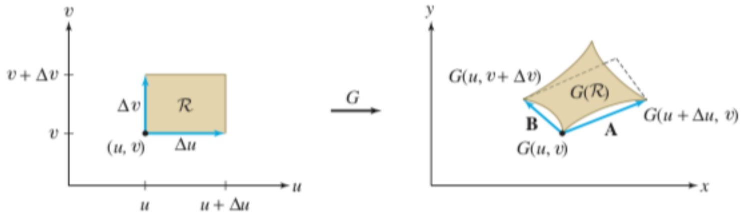
(i) If $A = a\vec{i} + b\vec{j}$ and $B = c\vec{i} + d\vec{j}$, then the area of the parallelogram spanned by A and B is the absolute value of det $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = |ad-bc|$.

(ii) For a function $f(u,v)$ whose partial derivatives exist:

$$f(u+\Delta u, v) - f(u,v) \approx \Delta u\frac{\partial f}{\partial u} \ \text{ and } \ f(u, v+\Delta v) - f(u,v) \approx \Delta v\frac{\partial f}{\partial v},$$

when Δu and Δv are small. This follows, since for example, $\frac{\partial f}{\partial u} \approx \frac{f(u+\Delta u,v)-f(u,v)}{\Delta u}$.

Now, G transforms the rectangle with area $\Delta u \Delta v$ to the curvilinear rectangle shown below:



In the Riemann sums of the double integral in x and y over the region $G(\mathcal{R})$, we may use the parallelogram P spanned by spanned by the vectors $\mathbf{A}$ and $\mathbf{B}$ as small portions of area dA. Note that

$$\begin{aligned}\mathbf{A} &= (x(u+\Delta u, v) - x(u,v))\ \vec{i} + (y(u+\Delta u, v) - y(u,v))\ \vec{j} \\ &\approx \Delta u \frac{\partial x}{\partial u}\ \vec{i} + \Delta v \frac{\partial x}{\partial v}\ \vec{j}.\end{aligned}$$

Similarly: $\mathbf{B} \approx \Delta v \frac{\partial x}{\partial v}\ \vec{i} + \Delta v \frac{\partial y}{\partial v}\ \vec{j}$. Therefore:

$$\begin{aligned} dA &\approx \text{area}(\mathcal{R}) \\ &\approx \text{area}(P) \\ &\approx \left| \det \begin{pmatrix} \Delta u \frac{\partial x}{\partial u} & \Delta v \frac{\partial x}{\partial v} \\ \Delta u \frac{\partial y}{\partial u} & \Delta v \frac{\partial y}{\partial v} \end{pmatrix} \right| \\ &= \left| \Delta u \Delta v \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \Delta u \Delta v \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| \\ &= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v. \end{aligned}$$

Thus, the Riemann sum: $\sum_i \sum_j f(x_i, y_j)\ dA$ in xy-coordinates is approximately the Riemann sum

$$\sum_i \sum_j f(x(u_j, v_j), y(u_i, v_j)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v,$$

in uv-coordinates. Passing to the limit as the units of area tend to zero gives the change of variables formula.

We ended class by discussing improper double integrals, first recalling familiar cases for single improper integrals. As in the single variable case, we noted improper integrals occur either when the integrand is not defined on boundary points of the domain of integration, is unbounded on the domain of integration, or when the domain of integration is infinite. We illustrated these ideas with the following examples.

Example 1. $\int_0^1 \frac{1}{\sqrt{x}}\ dx$. Note that $f(x)$ is unbounded on $[0,1]$, but $\lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\ dx = 2$ exists, so the original integral exists, or *converges* to 2.

Example 2. $\int_1^\infty e^{-2x}\ dx$. Note that the domain of integration is unbounded, but $\lim_{b\to\infty} \int_1^b e^{-2x}\ dx = \frac{1}{2e^2}$ exists, so the original integral converges.

Example 3. $\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA$, where $D: 0 \le x^2+y^2 \le 1$. Note that the integrand $f(x,y) = \frac{1}{\sqrt{1-x^2-y^2}}$ approaches infinity as points in the interior of the disk approach the circle $x^2+y^2 = 1$. Thus the integrand is unbounded on the domain of integration. Let D_a denote the disk $0 \le x^2+y^2 \le a^2$, with $0 < a < 1$. If the $\lim_{a\to 1} \int\int_{D_\epsilon} f(x,y)\ dA$ exists, then it will converge to the original integral, i.e.,

$$\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA = \lim_{a\to 1} \int\int_{D_a} \frac{1}{\sqrt{1-x^2-y^2}}\ dA.$$

We can use polar coordinates:

$$\begin{aligned}\int\int_{D_a} \frac{1}{\sqrt{-1-x^2-y^2}}\ dA &= \int_0^{2\pi}\int_0^a \frac{r}{\sqrt{1-r^2}}\ drd\theta \\ &= \int_0^{2\pi} -\sqrt{1-r^2}\Big|_{r=0}^{r=a}\ dA \\ &= \int_0^{2\pi} -\sqrt{1-a^2}+1\ dr \\ &= 2\pi\cdot(-\sqrt{1-a^2}+1).\end{aligned}$$

Taking the limit as $a \to 1$, we get 2π. Thus:

$$\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA = 2\pi.$$

In other words, $\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA$ converges to 2π.

Example 4. $\int\int_D xye^{-x^2-y^2}\ dA$, where D is the first quadrant of $\mathbb{R}^2$.

Solution: In this case we can proceed as one might expect:

$$\int\int_D xye^{-x^2-y^2}\ dA = \lim_{a,b\to\infty}\int_0^b\int_0^a xye^{-x^2-y^2}\ dxdy.$$

Here's one way to evaluate the iterated integral:

$$\begin{aligned}\int_0^b\int_0^a xye^{-x^2-y^2}\ dxdy &= \int_0^b\int_0^a (xe^{-x^2})(ye^{-y^2})\ dxdy \\ &= \int_0^b ye^{-y^2}(\int_0^a xe^{-x^2}\ dx)\ dy \\ &= (\int_0^b ye^{-y^2}\ dy)\cdot(\int_0^a xe^{-x^2}\ dx).\end{aligned}$$

Calculating these integrals separately:

$$\begin{aligned}\int_0^b ye^{-y^2}\ dy &= -\frac{1}{2}e^{-y^2}\Big|_{y=0}^{y=b} \\ &= \frac{1}{2}(-e^{-b^2}+1).\end{aligned}$$

Similarly: $\int_0^a xe^{-x^2}\ dx = \frac{1}{2}(-e^{-a^2}+1)$. Thus:

$$\int_0^b\int_0^a xye^{-x^2-y^2}\ dxdy = \frac{1}{2}(-e^{-b^2}+1)\cdot\frac{1}{2}(-e^{-a^2}+1).$$

Passing to the limit at $a\to\infty$ and $b\to\infty$, we get:

$$\int\int_D xye^{-x^2-y^2}\ dA = \frac{1}{4}.$$

Wednesday, October 23. We began class by calculating the improper integral $\int_{-\infty}^{\infty} e^{-x^2}\ dx$ by calculating its square as a double integral over $\mathbb{R}^2$. We noted that the answer then yields $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}\ dx = 1$, a formula used when discussing normal distributions.

We then began a discussion of triple integrals.

A triple integral is an integral of the form $\int\int\int_B f(x,y,z)\ dV$, where B is a solid region contained in $\mathbb{R}^3$. The underlying idea for the definition of $\int\int\int_B f(x,y,z)\ dV$ is the same as we discussed for double integrals: First, partition the domain of integration B into small subregions - in this case solids - of a similar type.

Second, select a point from each small subregion and evaluate the function $f(x, y, z)$ at that point.

Third, multiply the value obtained in the second step by the size of the subregion the point was chosen from. In this case we are multiplying the function value by a small unit of volume.

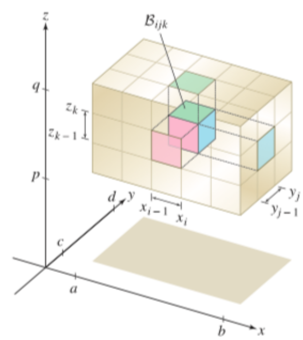
Fourth, add up the values from the previous step, thereby getting a Riemann sum.

Fifth, take a limit of the Riemann sums as the maximum volumes of the subregions in the partition go to zero.

Sixth, if the limit exists, we denote it $\int\int\int_B f(x, y, z)\ dV$.

As in previous discussions concerning double integrals, $\int\int\int_B f(x, y, z)\ dV$ is a quantity that depends upon $f(x, y, z)$ and the geometry of B and does not depend upon the coordinate system used to describe B or used to calculate $\int\int\int_B f(x, y, z)\ dV$.

As expected, we have various versions Fubini's Theorem for triple integrals. If B is a rectangular box, and we use rectangular coordinates, our Riemann sums look something like this, which justifies Fubini's Theorem:



$$S_{N,M,L} = \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{L} f(P_{ijk})\,\Delta V_{ijk}$$

Fubini's Theorem for rectangular boxes. Suppose $B = [a, b] \times [c, d] \times [p, q]$ is a rectangular box in $\mathbb{R}^3$ and $f(x, y, z)$ is continuous on B. Then:

$$\int\int\int_B f(x, y, z)\ dV = \int_a^b \int_c^d \int_p^q f(x, y, z)\ dzdydx.$$

Moreover, $\int\int\int_B f(x, y, z)\ dV$ can be calculated in any one of the five remaining ways to permute the order of integration. For example:

$$\begin{aligned}\int\int\int_B f(x, y, z)\ dV &= \int_c^d \int_p^q \int_a^b f(x, y, z)\ dxdzdy \\ &= \int_p^q \int_a^b \int_c^d f(x, y, z)\ dydxdz.\end{aligned}$$

Example 1. Calculate $\int\int\int_B x^2 + 2yz\ dV$, where $B = [0,1] \times [-1,0] \times [1,2]$.

Solution: Applying Fubini's Theorem,

$$\begin{aligned}
\int\int\int_B x^2 + 2yz\ dV &= \int_0^1 \int_{-1}^0 \int_1^2 (x^2 + 2yz)\ dzdydx \\
&= \int_0^1 \int_{-1}^0 (x^2 z + yz^2)_{z=1}^{z=2}\ dydx \\
&= \int_0^1 \int_{-1}^0 (2x^2 + 4y) - (x^2 + y)\ dydz \\
&= \int_0^1 \int_{-1}^0 x^2 + 3y\ dydx \\
&= \int_0^1 (x^2 y + \frac{3}{2} y^2)_{y=-1}^{y=0}\ dx \\
&= \int_0^1 (x^2 - \frac{3}{2})\ dx \\
&= (\frac{x^3}{3} - \frac{3x}{2})_{x=0}^{x=1} \\
&= \frac{1}{3} - \frac{3}{2} = -\frac{7}{6}.
\end{aligned}$$

Integrating in a different order we get:

$$\begin{aligned}
\int\int\int_B x^2 + 2yz\ dV &= \int_{-1}^0 \int_1^2 \int_0^1 (x^2 + 2yz)\ dxdzdy \\
&= \int_{-1}^0 \int_1^2 (\frac{x^3}{3} + 2xyz)_{x=0}^{x=1}\ dzdy \\
&= \int_{-1}^0 \int_1^2 (\frac{1}{3} + 2yz)\ dzdy \\
&= \int_{-1}^0 (\frac{z}{3} + yz^2)_{z=1}^{z=2}\ dy \\
&= \int_{-1}^0 \frac{1}{3} + 3y\ dy \\
&= (\frac{y}{3} + \frac{3y^2}{2})_{y=-1}^{y=0} \\
&= 0 - (-\frac{1}{3} + \frac{3}{2}) = -\frac{7}{6}.
\end{aligned}$$

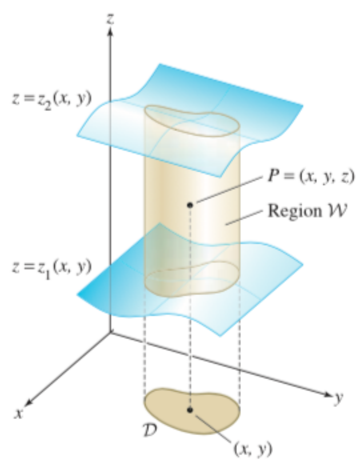
Fundamental Fact: $\int\int\int_B\ dV = \text{vol}(B)$. This follows, since in the Riemann sums approximating $\int\int\int_B\ dV$, we are just adding volumes of small solids covering B, so that each Reimann sum provides a better approximation to the volume of B and by passing to the limit, we obtain the volume of B.

Example 2. Let B denote the box in Example 1. Then:

$$\begin{aligned}
\int\int\int_B \, dV &= \int_0^1 \int_{-1}^0 \int_1^2 dzdydx \\
&= \int_0^1 \int_{-1}^0 (2-1)\, dydx \\
&= \int_0^1 \int_{-1}^0 \, dydx \\
&= \int_0^1 (0-(-1))\, dx \\
&= \int_0^1 \, dx \\
&= 1. \\
&= \mathrm{vol}(B)
\end{aligned}$$

as expected.

What about more general domains of integration? Suppose we have a region $\mathcal{W} \subseteq \mathbb{R}^3$ defined as all points (x, y, z) in $\mathbb{R}^3$ such that $(x, y) \in D$ and $z_1(x, y) \le z \le z_2(x, y)$, as pictured below.

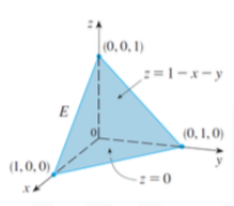


We have a corresponding version of Fubini's Theorem:

$$\int\int\int_{\mathcal{W}} f(x, y, z)\, dV = \int\int_D \left\{ \int_{z(_1x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right\} dA.$$

Note that if we calculate the integral in the brackets by integrating with respect to z, and then evaluating z at $z_2(x, y)$ and $z_1(x, y)$ and subtracting, we then have a double integral of a function in x and y only over the domain $D \subseteq \mathbb{R}^2$. In this version of Fubini's Theorem, we can reduce a triple to a double integral. In the case above, we call W a z-simple region.

Example 3. Calculate $\int\int\int_B e^{x+y+z}\, dV$ for B the solid tetrahedron below:



Solution. Regarding B as a z-simple region, we have that $0 \le z \le 1 - x - y$ and D is the triangle $0 \le x \le 1$, $0 \le y \le 1 - x$. Thus:

$$\begin{aligned}
\int\int\int_B e^{x+y+z}\, dV &= \int\int_D \{ \int_0^{1-x-y} e^{x+y+z}\, dz \}\, dA. \\
&= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^{x+y+z}\, dzdydx \\
&= \int_0^1 \int_0^{1-x} e^{x+y+z} \Big|_{z=0}^{z=1-x-y} dydx \\
&= \int_0^1 \int_0^{1-x} (e^1 - e^{x+y})\, dydx \\
&= \int_0^1 (ey - e^{x+y})_{y=0}^{y=1-x}\, dx \\
&= \int_0^1 (e(1-x) - e) - (0 - e^x)\, dx \\
&= \int_0^1 (-ex + e^x)\, dx \\
&= (-\frac{e}{2}x^2 + e^x)_0^1 \\
&= (-\frac{e}{2} + e) - (0 + 1) \\
&= \frac{e}{2} - 1.
\end{aligned}$$

We then noted the following. Suppose B is a z-simple region defined over the domain $D \subseteq \mathbb{R}^2$. If D is a region of Type 1, say $c(x) \le y \le d(x)$, $a \le x \le b$, then

$$\int\int\int_{\mathcal{B}} f(x,y,z)\, dV = \int_a^b \int_{c(x)}^{d(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz\, dy\, dx,$$

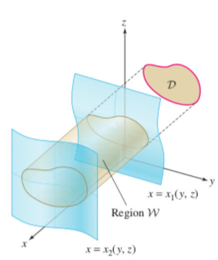
while if D is a region of Type 3 described as: $a(y) \le x \le b(y)$ and $c \le y \le d$, then

$$\int\int\int_{\mathcal{B}} f(x,y,z)\, dV = \int_c^d \int_{a(y)}^{b(y)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz\, dx\, dy.$$

We also noted that the one may have x-simple and y-simple regions, where in the first case, B is bounded in the front and back by functions $x = a(y,z)$ and $x = b(y,z)$, while in the second case, B is bounded to the left and right by functions $y = c(x,z)$ and $y = d(x,z)$, with the usual orientation of the x, y, z-axes. In our MW text, the z-simple, x-simple, and y-simple regions are called regions of Type I, II, or III, respectively.

Friday, October 25. We continued with our discussion of triple integrals, first reviewing the notion of z-simple regions discussed in the previous lecture, and then the consideration of triple integrals over regions that are not z-simple, but those that live above or below the yz-plane or xz-plane.

We have two other types of simple solids. An x-simple solid has the form $x_1(y,z) \le x \le x_2(y,z)$, $(y,z) \in D$, with D in the yz-plane.



By Fubini's Theorem:

$$\int\int\int_{\mathcal{W}} f\ dV = \int\int_{D} \{ \int_{x_1(y,z)}^{x_2(y,z)} f\ dx \}\ dA.$$

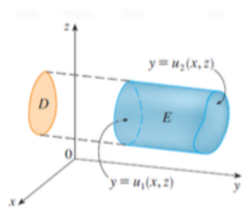
If D is described as: $a(z) \leq y \leq b(z)$ and $c \leq z \leq d$, then

$$\int\int\int_{\mathcal{W}} f\ dV = \int_c^d \int_{a(z)}^{b(z)} \int_{x_1(y,z)}^{x_2(y,z)} f\ dxdydz.$$

While if D is described as: $c(y) \leq z \leq d(y)$ and $a \leq y \leq b$, then

$$\int\int\int_{\mathcal{W}} f\ dV = \int_a^b \int_{c(y)}^{d(y)} \int_{x_1(y,z)}^{x_2(y,z)} f\ dxddzdy.$$

A y-simple solid E has the form $u_1(x,z) \leq y \leq u_2(x,z)$, $(y,z) \in D$, with D in the xz-plane.



By Fubini's Theorem:

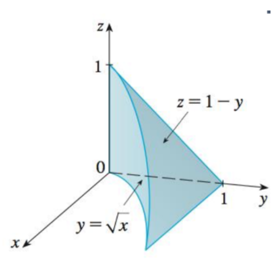
$$\int\int\int_{E} f\ dV = \int\int_{D} \{ \int_{u_1(x,z)}^{u_2(x,z)} f\ dx \}\ dA.$$

which can take the form

$$\int_c^d \int_{a(x)}^{b(x)} \int_{u_1(x,z)}^{u_2(x,z)} f\ dydzdx \quad \text{or} \quad \int_a^b \int_{c(z)}^{d(z)} \int_{u_1(x,z)}^{u_2(x,z)} f\ dydxdz$$

We then calculates the following example in two ways, first as a z-simple region, then as an x-simple region.

Example 1. Calculate $\int\int\int_B x\ dV$ for B



Solution

$$\begin{aligned}
\int\int\int_B x\ dV &= \int\int_D \{\int_0^{1-y} x\ dz\}\ dA \\
&= \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} x\ dzdydx \\
&= \int_0^1 \int_{\sqrt{x}}^1 xz\Big|_{z=0}^{z=1-y}\ dydx \\
&= \int_0^1 \int_{\sqrt{x}}^1 x(1-y)\ dydx \\
&= \int_0^1 x(y - \frac{y^2}{2})_{y=\sqrt{x}}^{y=1}\ dx \\
&= \int_0^1 x(\frac{1}{2} - \sqrt{x} + \frac{x}{2})\ dx \\
&= (\frac{x^2}{4} - \frac{2}{5}x^{\frac{5}{2}} + \frac{x^3}{6})_0^1 \\
&= \frac{1}{4} - \frac{2}{5} + \frac{1}{6} \\
&= \frac{1}{60}
\end{aligned}$$

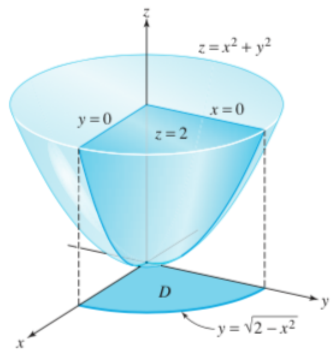
To calculate Calculate $\int\int\int_B x\ dV$ for B above, viewed as a x-simple surface, we first notice that if we move inside B in the direction of the x-axis, we get $0 \le x \le y^2$. The projection of B onto the yz-plane is the area in the first quadrant below the line $z = 1 - y$. Thus, we have

$$\begin{aligned}
\int\int\int_B x\ dV &= \int_0^1 \int_0^{1-y} \int_0^{y^2} x\ dx\ dz\ dy \\
&= \int_0^1 \int_0^{1-y} \{\frac{x^2}{2}\}_{x=0}^{x=y^2}\ dz\ dy \\
&= \int_0^1 \int_0^{1-y} \frac{y^4}{2}\ dz\ dy \\
&= \int_0^2 \{\frac{y^4}{2} z\}_{z=0}^{z=1-y}\ dy \\
&= \int_0^1 \frac{y^4}{2} - \frac{y^5}{2}\ dy \\
&= \frac{1}{10} - \frac{1}{12} \\
&= \frac{1}{60}.
\end{aligned}$$

We then recorded the two expected facts:

(i) $\int\int\int_B\ dV = \text{volume}(B)$.
(ii) The average value of $f(x, y, z)$ over B is: $\frac{1}{\text{vol}(B)} \int\int\int_B f(x, y, z)\ dV$.

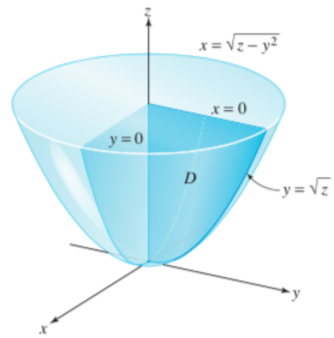
We then had the class set up the integral $\int\int\int_B x\ dV$ for B viewing B below as a z-simple region and as an x-simple region. Details of the calculation were not done in class.



Solution. Viewing B as a z-simple region, we have

$$
\begin{aligned}
\int\int\int_B x\ dV &= \int\int_D \int_{x^2+y^2}^{2} x\ dz\ dA \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x\ dzdydx \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} xz\Big|_{z=x^2+y^2}^{z=2} dydx \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} x(2-x^2-y^2)dydx \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r\cos(\theta)(2-r^2)r\ drd\theta \\
&= \int_0^{\frac{\pi}{2}} \cos(\theta) \int_0^{\sqrt{2}} 2r^2 - r^4\ drd\theta \\
&= \int_0^{\frac{\pi}{2}} \cos(\theta)\cdot(\frac{2r^3}{3} - \frac{r^5}{5})_{r=0}^{r=\sqrt{2}}\ d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos(\theta)\cdot(\frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{5})\ d\theta \\
&= \frac{8\sqrt{2}}{15} \int_0^{\frac{\pi}{2}} \cos(\theta)\ d\theta \\
&= \frac{8\sqrt{2}}{15}.
\end{aligned}
$$

For the same integral integrating x first.

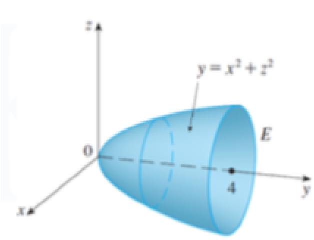


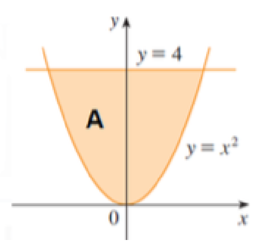
Note that $0 \leq x \leq \sqrt{z - y^2}$ and the domain D in the yz-plane is bounded below by the parabola $z = y^2$ and above by $z = 2$. Thus

$$
\begin{aligned}
\int\int\int_B x \; dV &= \int_0^{\sqrt{2}} \int_{y^2}^{2} \int_0^{\sqrt{z-y^2}} x \; dxdzdy \\
&= \frac{1}{2} \int_0^{\sqrt{2}} \int_{y^2}^{2} z - y^2 \; dzdy \\
&= \frac{1}{2} \int_0^{\sqrt{2}} \left(\frac{z^2}{2} - y^2 z\right)_{z=y^2}^{z=2} \; dy \\
&= \frac{1}{2} \int_0^{\sqrt{2}} (2 - 2y^2) - \left(\frac{y^4}{2} - y^4\right) \; dy \\
&= \frac{1}{2} \int_0^{\sqrt{2}} 2 - 2y^2 + \frac{y^4}{2} \; dy \\
&= \frac{1}{2}\left(2y - \frac{2y^3}{3} + \frac{y^5}{10}\right)_0^{\sqrt{2}} \\
&= \frac{1}{2}\left\{2\sqrt{2} - \frac{4\sqrt{2}}{3} + \frac{4\sqrt{2}}{10}\right\} \\
&= \frac{8\sqrt{2}}{15}.
\end{aligned}
$$

We then worked the following example:

Example. This example shows how changing the order of integration can simplify the integration. Consider $\int\int\int_E \sqrt{x^2 + z^2} \; dV$ for the solid E pictured on the left



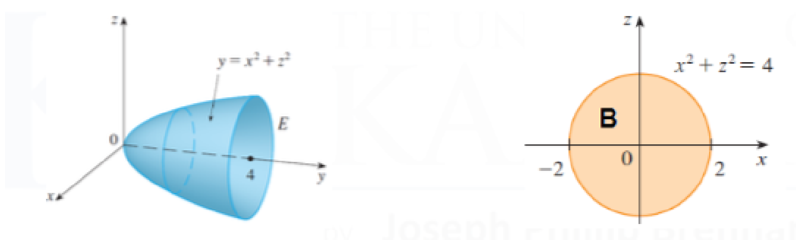


Regarding E as a z-simple region, with domain of integration A, we have

$$\begin{aligned}\int\int\int_E \sqrt{x^2+z^2}\ dV &= \int\int_A \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dz\ dA \\ &= \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dzdyx \\ &= \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dzdxdy,\end{aligned}$$

which is difficult to integrate

On the other hand, regarding E as a y-simple region with domain of integration , we tegrate with respect to y first, to obtain

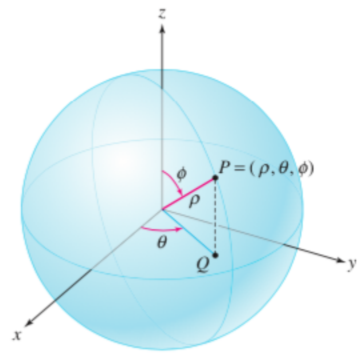


$$\begin{aligned}\int\int\int_E \sqrt{x^2+z^2}\ dV &= \int\int_B \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\ dy\ dB \\ &= \int\int_B 4\sqrt{x^2+z^2} - (x^2+z^2)\sqrt{x^2+z^2}\ dB \\ &= \int_0^{2\pi}\int_0^2 (4r-r^3)r\ drd\theta \\ &= 2\pi\int_0^2 4r^2 - r^4\ dr \\ &= 2\pi(\frac{32}{3}-\frac{32}{5}) \\ &= \frac{128\pi}{15}.\end{aligned}$$

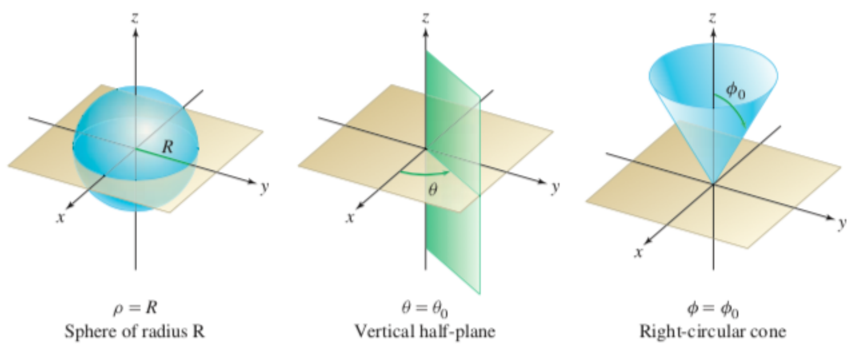
We ended class by considering the integral $\int\int\int_B \sqrt{x^2+y^2+z^2}\ dV$, where B is the solid sphere of radius one centered at the origin. We noted that this requires a three dimensional version of polar coordinates, and suggested that a substitution using spherical coordinates should be the analogous procedure. Examples of this type will be discussed in the next lecture.

Monday, October 28. In today's lecture, Jake began a discussion of evaluating triple integrals using *spherical coordinates* as a means to calculatingintegrals of the form $\int\int\int_B \sqrt{x^2+y^2+z^2}\ dV$, where $B \subseteq \mathbb{R}^3$ is the solid sphere of radius one centered at the originn.One begins by observing that every point in $\mathbb{R}^3$ lies on a sphere of radius ρ centered at (0,0,0) and thus can be expressed in terms of spherical coordinates, ρ, ϕ, θ.

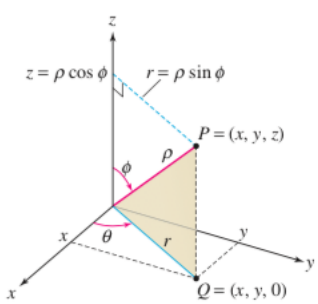
Here is typical point P using spherical coordinates



Note: $P = (\rho, \phi, \theta)$, with $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. If we set each of the spherical coordinate equal to a constant, we get:



We then showed the relation between the spherical coordinates and the rectangular coordinates of P.



Note:

$$x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta), y = r\sin(\theta) = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi).$$

It follows from these equations that the expression $x^2 + y^2 + z^2$, in spherical coordinates, becomes ρ^2. We will use this fact often. Writing spherical coordinates in terms of rectangular coordinates.

$\rho = \sqrt{x^2 + y^2 + z^2}$
$\tan(\theta) = \frac{y}{x}$, so $\theta = \tan^{-1}(\frac{y}{x})$.
$\cos(\phi) = \frac{z}{\rho}$, so $\phi = \cos^{-1}(\frac{z}{\rho})$.

Example 1. Find the rectangular coordinates of the point $P = (\rho, \phi, \theta) = (3, \frac{\pi}{3}, \frac{\pi}{4})$.

Solution

$$x = 3\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) = 3\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{3\sqrt{6}}{4}$$
$$y = 3\sin(\frac{\pi}{3})\sin(\frac{\pi}{4}) = 3\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{3\sqrt{6}}{4}$$
$$z = 3\cos(\frac{\pi}{3}) = 3\cdot\frac{1}{2} = \frac{3}{2}.$$

Example 2. Find the spherical coordinates of the point $P = (x, y, z) = (-1, 1, \sqrt{6})$.

Solution. $\rho = \sqrt{(-1)^2 + 1^2 + (\sqrt{6})^2} = \sqrt{8} = 2\sqrt{2}$. From $z = \rho\cos(\phi)$ we have $\sqrt{6} = 2\sqrt{2}\cos(\phi)$.Thus, $\cos(\phi) = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$. $\phi = \frac{\pi}{6}$. $\theta = \tan^{-1}(\frac{-1}{1}) = \tan^{-1}(-1) = \frac{3\pi}{4}$, since $(x, y) = (-1, 1)$. Thus, in spherical coordinates, $P = (2\sqrt{2}, \frac{\pi}{6}, \frac{3\pi}{4})$.

When then saw that to convert a triple integral $\int\int\int_B f(x, y, z)\ dV$ into an iterated integral involving spherical coordinates, there are two step involved. First, one describes B in terms of spherical coordinates, as follows: $\rho_1(x, y) \le \rho \le \rho_2(x, y)$, $\phi_1 \le \pi \le \phi_2$, $\theta_1 \le \theta \le \theta_2$. We then set

$$x = \rho\sin(\phi)\cos(\theta), y = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi) \quad \text{and} \quad dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta,$$

with the understanding that we will explain the equality $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$ in our next lecture. Thus, by Fubini's Theorem, we have

$$\int\int\int_B f(x, y, z)\ dV = \int_{\theta_1}^{\theta_2}\int_{\rho_1}^{\phi_2}\int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\ \rho^2\sin(\phi)\ d\rho d\phi d\theta.$$

We then worked the example mentioned at the outset.

Example 3. Calculate $\int\int\int_B \sqrt{x^2 + y^2 + z^2}\ dV$, where $B \subseteq \mathbb{R}^3$ is the solid sphere of radius one centered at the origin.

Solution. We first noted that in terms of spherical coordinates, B can be described by

$$0 \le \rho \le 1, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

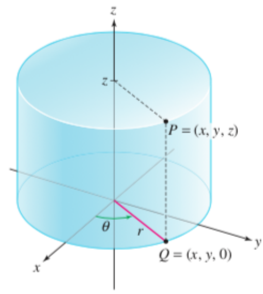
Thus,

$$\begin{aligned}
\int\int\int_B \sqrt{x^2 + y^2 + z^2}\ dV &= \int\int\int_B \sqrt{\rho^2\sin^2(\phi)\cos^2(\theta) + \rho^2\sin^2(\phi)\sin^2(\theta) + \rho^2\cos^2(\phi)}\ \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^3\sin(\phi)\ d\rho\ d\phi\ d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi}\sin(\phi)\ d\phi\ d\theta \\
&= \frac{2\pi}{4}\int_0^{\pi}\sin(\phi)\ d\phi \\
&= \frac{2\pi}{4}\cdot 2 \\
&= \pi.
\end{aligned}$$

Wednesday, October 30. Jake continued the discussion involving triple integral by introducing cylindrical coordinates as a means to simplify triple integral calculations. Below are relevant information and examples.

The key observation regarding polar coordinates is that every point in $\mathbb{R}^3$ lies on the top edge of a cylinder, which enables us to describe points in $\mathbb{R}^3$ in terms of cylindrical coordinates, noting that cylindrical coordinates are essentially like polar coordinates, though with the extra variable z.

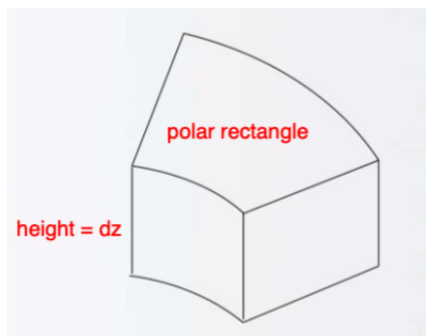
In cylindrical coordinates, $P = (r, \theta, z)$.



To transform a triple integral into cylindrical coordinates, we set:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z, \quad dV = r\ dz\ dr\ d\theta.$$

We can easily guess what the volume of a cylindrical wedge should be.



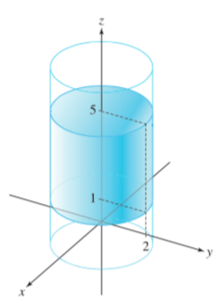
It should be approximately the area of the corresponding polar rectangle times Δz, the change in the z direction, i.e., the height of the cylindrical wedge. Thus, in cylindrical coordinates, small units of volume dV are approximately $(r\Delta r\Delta\theta)\cdot\Delta z = r\ \Delta z\ \Delta r\ \Delta\theta$. Using expressions of this type in the Riemann sum for $\int\int\int_B f(x,y,z)\ dV$ in cylindrical coordinates, we get the following version of Fubini's theorem:

Fubini's Theorem in Cylindrical Coordinates. Given a bounded region $B \subseteq \mathbb{R}^3$ described in cylindrical coordinates as: $z_1(r,\theta) \leq z_2(r,\theta); r_1(\theta) \leq r \leq r_2(\theta); \theta_1 \leq \theta \leq \theta_2$, and $f(x,y,z)$ continuous on B, we have:

$$\int\int\int_B f(x,y,z)\ dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos(\theta), r\sin(\theta), z)\ r\ dz\ dr\ d\theta.$$

We then worked some examples using cylindrical coordinates.

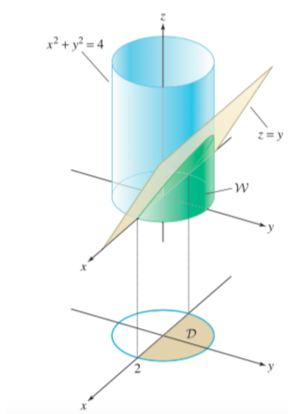
Example 2. Integrate $z\sqrt{x^2+y^2}$ over the cylinder B given below.



Solution: In cylindrical coordinates $B: 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, 1 \leq z \leq 5$. Thus,

$$\begin{aligned}
\int\int\int_B z\sqrt{x^2+y^2} &= \int_0^{2\pi}\int_0^2\int_1^5 z\sqrt{(r\cos(\theta))^2+(r\sin(\theta))^2}\ r\ dzdrd\theta \\
&= \int_1^5\int_0^{2\pi}\int_0^2 zr^2\ drd\theta dz \\
&= 2\pi\int_1^5\int_0^2 zr^2\ drdz \\
&= 2\pi\int_1^5 z(\frac{2^3}{3}-0)\ dz \\
&= \frac{16\pi}{3}\int_1^5 z\ dz \\
&= \frac{16\pi}{3}\{\frac{5^2}{2}-\frac{1}{2}\} \\
&= 64\pi.
\end{aligned}$$

Example 3. Calculate $\int\int\int_W z\ dV$ for W bounded by the cylinder $0 \leq x^2+y^2 \leq 4$ and the planes $z = y$, with $y \geq 0$.

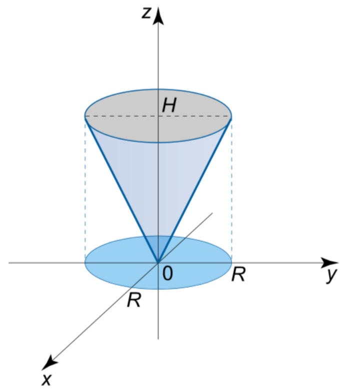


To describe W is cylindrical coordinates: $0 \leq z \leq y$, so $0 \leq z \leq r\sin(\theta)$. Since $y \geq 0$, the projection of W onto the xy-plane is the semi-circle D. Thus, $0 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.

$$\begin{aligned}
\int\int\int_W z\ dV &= \int_0^{\pi}\int_0^2\int_0^{r\sin(\theta)} z\ rdzdrd\theta \\
&= \frac{1}{2}\int_0^{\pi}\int_0^2 r^3\sin^2(\theta)\ drd\theta \\
&= \frac{1}{2}\int_0^{\pi}\frac{16}{4}\sin^2(\theta)\ d\theta \\
&= 2\int_0^{\pi}\frac{1}{2}-\frac{1}{2}\cos(2\theta)\ d\theta \\
&= 2\{\frac{\theta}{2}-\frac{1}{4}\sin(2\theta)\}_0^{\pi} \\
&= \pi.
\end{aligned}$$

Example 4. Find the volume of a cone with height H and radius R.

Solution: If we take $z = \frac{H}{R}\sqrt{x^2+y^2}$, we have the cone (can you see why?):



If we let B denote the solid cone, then z is bounded below by the cone and above by the plane $z = H$. In cylindrical coordinates, the cone is $z = \frac{H}{R} \cdot r$. Thus,

$$\begin{aligned}
\text{vol}(B) &= \int\int\int_B dV \\
&= \int_0^R \int_0^{2\pi} \int_{\frac{H}{R}r}^{H} r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^R rz\Big|_{z=\frac{H}{R}r}^{z=H} dr\ d\theta \\
&= \int_0^{2\pi} \int_0^R rH - \frac{H}{R}r^2\ dr\ d\theta \\
&= \int_0^{2\pi} (\frac{r^2}{2}H - \frac{H}{R}\cdot\frac{r^3}{3})\Big|_{r=0}^{r=R} d\theta \\
&= \int_0^{2\pi} \frac{R^2H}{2} - \frac{R^2H}{3}\ d\theta \\
&= \frac{R^2H}{6}\int_0^{2\pi} d\theta \\
&= \frac{R^2H}{6}\cdot 2\pi \\
&= \frac{\pi R^2 H}{3}.
\end{aligned}$$

Friday, November 1. The class worked on review problems for the second exam.

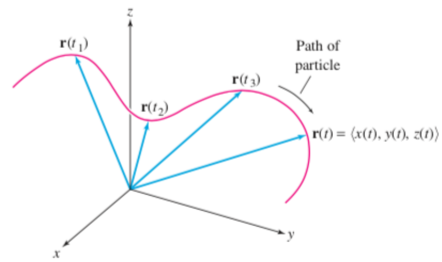
Monday, November 4. The class continued to work on review problems for the second exam.

Wednesday, November 6. Our next goal is to integrate along a curve in $\mathbb{R}^3$. For this, we began a discussion of vector valued functions.

Definition. A vector valued function is a function

$$\mathbf{r}(t) = x(t)i + y(t)j + z(t)k = (x(t), y(t)z(t)),$$

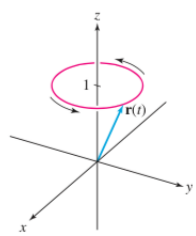
with t belonging to a subset of $\mathbb{R}$.



If the values of $\mathbf{r}(t)$ lie in the xy-plane, we write

$$\mathbf{r}(t) = (x(t), y(t)) = x(t)i + y(t)j.$$

The variable t is called the *parameter*. It is often convenient to think of t as time. The set of points traced out by $\mathbf{r}(t)$ is a *curve*. How the curve is traced out is called a *path*. For example, for the curve C, the circle of radius one, with center $(0, 0, 1)$



one can have several different paths:

$\mathbf{r}_1(t) = (\cos(t), \sin(t), 1)$, with $0 \leq t \leq 2\pi$ traces the curve once.

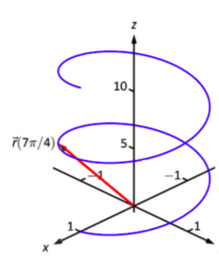
$\mathbf{r}_2(t) = (\cos(2t), \sin(2t), 1)$, with $0 \leq t \leq \pi$ traces once, but twice as fast.

$\mathbf{r}_3(t) = (\cos(t), \sin(t), 1)$, with $0 \leq t \leq 4\pi$, traces the curve twice, but at the same speed at $\mathbf{r}_1(t)$.

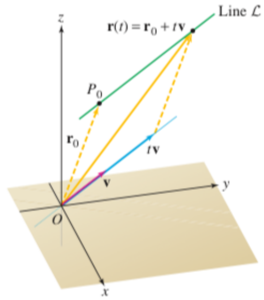
$\mathbf{r}_4(t) = (\cos(2\pi - t), \sin(2\pi - t), 1)$, traces the curve once, in the opposite direction of $\mathbf{r}_1(t)$.

The different paths $\mathbf{r}_i(t)$ above are referred to as different *parametrizations* of C. We then gave two more examples of curves in $\mathbb{R}^3$.

Example. The helix $\mathbf{r}(t) = (cos(t), \sin(t), t)$, $t \geq 0$.



Example. The line through a point $P_0 = (a, b, c)$ parallel to a given vector $\vec{v} = v_1 i + v_2 j + v_3 k$.



(B) Line through P_0 in the direction of $\mathbf{v}$

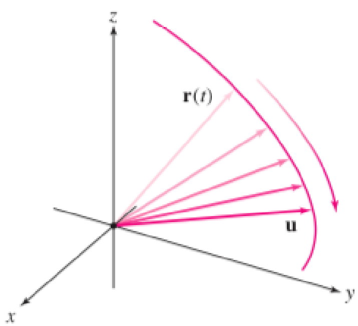
$$\mathbf{r}(t) = P_0 + t\vec{v} = (a + tv_1, b + tv_2, c + tv_3).$$

We then noted that limits and continuity for vector valued functions are defined in a similar way as for functions we have previously encountered.

Limits. For a vector valued function $\mathbf{r}(t)$ and a fixed vector $\vec{u} = u_1 i + u_2 j + u_3 k$, we write:

$$\lim_{t \to t_0} \mathbf{r}(t) = \vec{u} \quad \text{if} \quad \lim_{t \to t_0} ||\mathbf{r}(t) - \vec{u}|| = 0.$$

Note that this means the vectors $\mathbf{r}(t)$ get closer to the vector $\vec{u}$ as t approaches t_0.



DF **FIGURE 1** The vector-valued function $\mathbf{r}(t)$ approaches the vector $\mathbf{u}$ as $t \to t_0$.

This is equivalent to

$$\lim_{t \to t_0} x(t) = u_1$$
$$\lim_{t \to t_0} y(t) = u_2$$
$$\lim_{t \to t_0} z(t) = u_3.$$

To explain why this holds, we noted

$$\begin{aligned}\lim_{t \to t_0} ||\mathbf{r}(t) - \vec{u}|| &= \lim_{t \to t_0} \sqrt{(x(t) - u_1)^2 + (y(t) - u_2)^2 + (z(t) - u_3)^2} \\ &= \sqrt{(\lim_{t \to t_0} x(t) - u_1)^2 + (\lim_{t \to t_0} y(t) - u_2)^2 + (\lim_{t \to t_0} z(t) - u_3)^2}\end{aligned}$$

by continuity of the square root and square functions. If

$$\lim_{t\to t_0} x(t) = u_1$$
$$\lim_{t\to t_0} y(t) = u_2$$
$$\lim_{t\to t_0} z(t) = u_3,$$

then the limits under the radical are zero. Thus, $\lim_{t\to t_0} ||\mathbf{r}(t) - \vec{u}|| = 0$. i.e., $\lim_{t\to t_0} \mathbf{r}(t) = \vec{u}$.

Example. If $\mathbf{r}(t) = (5\cos(t), -3\sin(\frac{t}{2}), e^{3t+4})$, then

$$\begin{aligned}\lim_{t\to\pi} \mathbf{r}(t) &= (\lim_{t\to\pi} 5\cos(t), \lim_{t\to\pi} -3\sin(\frac{t}{2}), \lim_{t\to\pi} e^{3t+4})\\ &= (5\cos(\pi), -3\sin(\frac{\pi}{2}), e^{3\pi+4})\\ &= (-5, -3, e^{3\pi+4}).\end{aligned}$$

Continuity. $\mathbf{r}(t)$ is *continuous at* t_0 if $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. From the discussion of limits, we observed that this is equivalent to:

$$x(t), y(t), z(t) \text{ are all continuous at } t_0.$$

We then noted that differentiabiility is defined as expected.

Differentiability. $\mathbf{r}(t)$ is *differentiable* at t_0 if

$$\lim_{t\to h} \frac{1}{h} \cdot \{\mathbf{r}(t+h) - \mathbf{r}(t_0)\},$$

exists. Note this limit involves a scalar times a vector, and is a vector if the limit exists. If the limit exists, we write it as $\mathbf{r}'(t_0)$ or $\frac{d}{dt}\mathbf{r}(t)|_{t_0}$.

Fact: $\mathbf{r}'(t)$ is differentiable at t_0 exactly when $x(t), y(t), z(t)$ are all differentiable at t_0, in which case

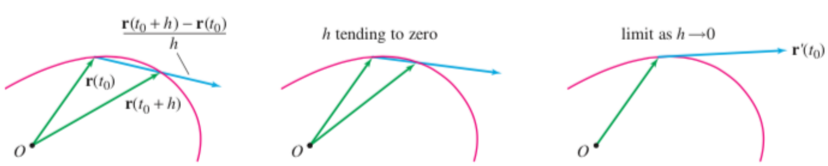
$$\mathbf{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0)).$$

This follows, since we may take limits coordinate-wise (and because the scalar $\frac{1}{h}$ can be moved inside of an ordered triple):

$$\begin{aligned}\lim_{t\to h} \frac{1}{h} \cdot \{\mathbf{r}(t+h) - \mathbf{r}(t_0)\} &= (\lim_{t\to h} \frac{x(t_0+h) - x(t_0)}{h}, \lim_{t\to h} \frac{y(t_0+h) - y(t_0)}{h}, \lim_{t\to h} \frac{z(t_0+h) - z(t_0)}{h})\\ &= (x'(t_0), y'(t_0), z'(t_0)).\end{aligned}$$

Example. Given $\mathbf{r}(t) = (5\cos(t), -3\sin(\frac{t}{2}), e^{3t+4})$, $\mathbf{r}'(t) = (-5\sin(t), -\frac{3}{2}\cos(\frac{t}{2}), 3e^{3t+4})$. Therefore: $\mathbf{r}\,'(\pi) = (-5\sin(\pi), \frac{3}{2}\cos(\frac{\pi}{2}), e^{3\pi+4}) = (0, 0, e^{3\pi+4})$.

We finished class by noting that the vector $\mathbf{r}'(t_0)$ is tangent to the curve $\mathbf{r}(t)$ at the point $P_0 = \mathbf{r}(t_0)$, as illustrated in the diagram below.



We then noted that we have versions of the familiar rules of differentiation for vector valued functions.

Properties of the derivative. Assuming differentiablity:

(i) $(\mathbf{r}(t) + \mathbf{r}s(t))' = \mathbf{r}'(t) + \mathbf{r}s'(t)$.
(ii) $(\lambda\mathbf{r}(t))' = \lambda\mathbf{r}'(t)$, for $\lambda \in \mathbb{R}$.
(iii) $(f(t) \cdot \mathbf{r}(t))' = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$, for $f(t)$ a scalar valued function.
(iv) $\mathbf{r}(g(t))' = g'(t)\mathbf{r}'(g(t))$, for $g(t)$, a scalar function.

(v) $(\mathbf{r}(t) \cdot \vec{s}(t))' = \mathbf{r}'(t) \cdot \vec{s}(t) + \mathbf{r}(t) \cdot \vec{s}'(t)$.
(vi) $(\mathbf{r}(t) \times \vec{s}(t))' = \mathbf{r}\ '(t) \times \vec{s}(t) + \mathbf{r}(t) \times \vec{s}\ '(t)$.

We then discussed the length of a path or curve.

Definition. Suppose $\mathbf{r}(t)$ is differentiable and $\mathbf{r}'(t)$ is continuous on $[a, b]$. Then the length of the path from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ is given by

$$s = \int_a^b ||\mathbf{r}'(t)||\ dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt.$$

The parameter s is called *arc length*. We can keep track of the arc length as we move along the path by considering the function:

$$s(t) = \int_a^t ||\mathbf{r}'(u)||\ du,$$

for $a \le t \le b$. We noted that s gives the length of the path $\mathbf{r}(t)$ for the range $a \le t \le b$, and that this is also the length of the corresponding curve, under an additional hypothesis.

IMPORTANT POINTS. (a) If $\mathbf{r}(t)$ is 1-1, then the arc length of the path equals the length of the curve traced out by $\mathbf{r}(t)$.

(b) The length of the curve traced out by $\mathbf{r}(t)$ is *independent of the parametrization.*

Example. We then revisited the examples from the previous lecture of the circle of radius one centered at (0,0,1) and its four parametrizations.

(i) $\mathbf{r}_1(t) = (\cos(t), \sin(t), 1)$, with $0 \le t \le 2\pi$ traces the curve once.
(ii) $\mathbf{r}_2(t) = (\cos(2t), \sin(2t), 1)$, with $0 \le t \le \pi$ traces once, but twice as fast.
(ii) $\mathbf{r}_3(t) = (\cos(t), \sin(t), 1)$, with $0 \le t \le 4\pi$, traces the curve twice, but at the same speed at $\mathbf{r}_1(t)$.
(iv) $\mathbf{r}_4(t) = (\cos(2\pi - t, \sin(2\pi - t), 1)$, traces the curve, once in reverse order.

We expect the lengths of (i), (i), (iv) to be 2π and the length of the path (iii) to be 4π.

For $\mathbf{r}_1(t) : ||\mathbf{r}_1'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 0} = 1$. Thus:

$$s = \int_0^{2\pi} ||\mathbf{r}_1'(t)||\ dt = \int_0^{2\pi} 1\ dt = 2\pi.$$

Note that $\mathbf{r}_1(t)$ is 1-1 on $[0, 2\pi]$, so this gives the expected length of the curve.

For $\mathbf{r}_2(t) : ||\mathbf{r}_2'(t)|| = \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2 + 0} = 2$. Thus:

$$s = \int_0^{\pi} ||\mathbf{r}_2'(t)||\ dt = \int_0^{\pi} 2\ dt = 2\pi.$$

Note that $\mathbf{r}_2(t)$ is also 1-1, so we get that the length of the path equals the length of the curve. Note also that the previous two parametrizations are the different, but yield the same arc length.

For $\mathbf{r}_3(t) : ||\mathbf{r}_3'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 0} = 1$. Thus

$$s = \int_0^{4\pi} ||\mathbf{r}_3'(t)||\ dt = \int_0^{4\pi} 1\ dt = 4\pi.$$

Note here that the path traces the curve twice, so the length of the path is 4π, while the length of the curve is 2π. In this case, $\mathbf{r}_3(t)$ is NOT 1-1 on the interval $[0, 4\pi]$, which explains why the length of the path differs from the length of the curve.

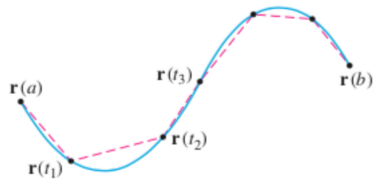
For $\mathbf{r}_4(t) : ||\mathbf{r}_4'(t)|| = \sqrt{(\cos(2\pi - t))^2 + (-\sin(2\pi - t))^2 + 0^2} = 1$. Thus

$$s = \int_0^{\pi} ||\mathbf{r}_4'(t)||\ dt = \int_0^{2\pi} 1\ dt = 2\pi,$$

as expected.

Friday, November 8. We reviewed the definition of arc length and the arc length function, noting that if we think of t as time, then $s(t) = \int_a^t ||\mathbf{r}'(t)||\ dt$ gives the distance traveled in time t along the path described by $\mathbf{r}(t)$, so that $s'(t) = ||\mathbf{r}'(t)||$ denotes the speed. Thus, the *velocity vector* $\mathbf{r}'(t)$ points in the direction of a point traveling along the curve (since it is tangent to the curve) and the length of the velocity vector gives the speed at time t.

We then gave a heuristic description accounting for the formula for the arc length along a segment of a curve, say C is given by $\mathbf{r}(t)$, with $a \le t \le b$. Partition the interval $[a, b] : a = t_1 < t_1 \cdots < t_n = b$ so that $t_{i+1} - t_i = \Delta t$ is small.



Create a polygonal path whose endpoints are the $\mathbf{r}(t_i)$. The length of each line segment in the path is $||\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)||$. We have:

$$||\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)|| = \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2}$$

For small values of Δt:

$$\begin{aligned} x(t_{i+1}) - x(t_i) &\approx x'(t_i)\Delta t \\ y(t_{i+1}) - y(t_i) &\approx y'(t_i)\Delta t \\ z(t_{i+1}) - z(t_i) &\approx z'(t_i)\Delta t \end{aligned}$$

Thus:

$$\begin{aligned} ||\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)|| &= \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2} \\ &\approx \sqrt{(x'(t_i)\Delta t)^2 + (y'(t_i)\Delta t)^2 + (z'(t_i)\Delta t)^2} \\ &= \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\ \Delta t \\ &= ||\mathbf{r}'(t_i)||\ \Delta t \end{aligned}$$

Summing these expressions we get an approximation of the arc length on the one hand, and a Riemann sum for $\int_a^b ||\mathbf{r}'(t)||\ dt$ on the other hand. Passing to the limit as $\Delta t \to 0$ gives the arc length formula.

We then turned to the definition of $\int_C f(x, y, z)\ ds$, the line integral of the scalar function $f(x, y, z)$ along the curve C. We proceed as before, as with all previous types of integration.

Step 1. Subdivide C into finitely many smaller curves C_i of the same length Δs.

Step 2. Choose a point (x_i, y_i, z_i) from the component C_i.

Step 3. Multiply $f(x_i, y, z_i)$ by the size of each C_i to get $f(x_i, y_i, z_i)\ \Delta s$.

Step 4. Add the products in Step 3 to get the Riemann sum: $\Sigma_i f(x_i, y_i, z_i)\Delta s$.

Step 5. Take the limit of the Riemann sums as $\Delta s \to 0$, to get:

$$\int_C f(x, y, z)\ ds,$$

the *line (or path) integral over* $f(x, y, z)$ *over* C. This integral is sometimes called the *the line integral of* $f(x, y, z)$ *with respect to arc length.*

We must use the parametrization $\mathbf{r}(t)$ of C to calculate $\int_C f(x, y, z)\ ds$. From our discussion of arc length, we have that small portions of length Δs along the curve are approximated by $||\mathbf{r}'(t_i)||\Delta t$, where $(x_i, y_i, z_i) = \mathbf{r}(t_i)$. Now use

$$\Sigma_i f(x_i, y_i, z_i)||\mathbf{r}'(t_i)||\ \Delta t$$

in the Riemann sum (Step 4) above. Taking the limit as $t \to 0$ we get:

$$\begin{aligned}\int_C f(x,y,z)\ ds &= \int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||\ dt \\ &= \int_a^b f(x(t),y(t),z(t))||\mathbf{r}'(t)||\ dt.\end{aligned}$$

A word of caution: Depending upon the context, one will either calculate a *path integral*, if the parametrization is given or one may have to choose a parametrization to calculate a line integral. Moreover, if the path $\mathbf{r}(t)$ doubles back on itself, or repeats portions of the curve with non-zero length, this will be reflected in $\int_a^b f(\mathbf{r}(t))\ ||\mathbf{r}'(t)||\ dt$. Note that:

$$\int_C ds = \int_a^b ||\mathbf{r}'(t)||\ dt,$$

is the arc length of C, which is what we expect.

Example. Calculate $\int_C xe^{z^2}\ ds$, for C the line segment give by $\mathbf{r}(t) = (t, 2-t, t)$, $0 \le t \le 1$.

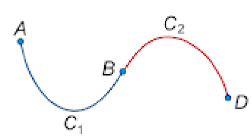
Solution. $\mathbf{r}'(t) = (1,-1,1)$, so $||\mathbf{r}'(t)|| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$. Since $x = t = z$ along the curve, we have

$$\begin{aligned}\int_C xe^{z^2}\ ds &= \int_0^1 te^{t^2}\ \sqrt{3}\ dt \\ &= \sqrt{3}\cdot\frac{1}{2}e^{t^2}\Big|_{t=0}^{t=1} \\ &= \frac{\sqrt{3}}{2}(e-1).\end{aligned}$$

We then discussed the following properties of line integrals.

Properties of line integrals. Assuming the line integrals below exist, we have

(i) $\int_C f(x,y,z)+g(x,y,z)\ ds = \int_C f(x,y,z)\ ds + \int_C g(x,y,z)\ ds$.
(ii) $\int_C \lambda f(x,y,z)\ ds = \lambda \int_C f(x,y,z)\ ds$, $\lambda \in \mathbb{R}$.
(iii) $\int_C f(x,y,z)\ ds = \int_{C_1} f(x,y,z)\ ds + \int_{C_2} f(x,y,z)\ ds$.



(iv) $\int_C f(x,y,z)\ ds$ is independent of the parametrization, if the parametrization is 1-1.

Example. This example illustrates property (iii). Let C be the triangle in $\mathbb{R}^3$ with vertices (2,0,0), (0,2,0), (0,0,2). The C has three components, C_1, the line segment from (2,0,0), C_2, the line segment from (0,2,0) to (0,0,2), and C_3, then line segment from (0,0,2), to (2,0,0). By property (iii),

$$\int_C xy+z^2\ ds = \int_{C_1} xy+z^2\ ds + \int_{C_2} xy+z^2\ ds + \int_{C_3} xy+z^2\ ds.$$

The parametrizations for these curves, respectively, are:

$$\begin{aligned}\mathbf{r}_1(t) &= (2-2t, 2t, 0),\ \ 0\le t\le 1 \\ \mathbf{r}_2(t) &= (0, 2-2t, 2t),\ \ 0\le t\le 1 \\ \mathbf{r}_3(t) &= (2t, 0, 2-2t),\ \ 0\le t\le 1.\end{aligned}$$

It is easy to check that $||\mathbf{r}_i(t)|| = 8$, for all i. Thus,

$$\begin{aligned}
\int_{C_1} xy + z^2 \ ds &= \int_0^1 (2-2t)2t + 0^2 \ \sqrt{8} \ dt \\
&= \frac{2\sqrt{8}}{3} \\
\int_{C_2} xy + z^2 \ ds &= \int_0^1 0(2-2t) + (2t)^2 \ \sqrt{8} \ dt \\
&= \frac{4\sqrt{8}}{3} \\
\int_{C_3} xy + z^2 \ ds &= \int_0^1 2t \cdot 0 + (2-2t)^2 \ \sqrt{8} \ dt \\
&= \frac{4\sqrt{8}}{3}.
\end{aligned}$$

Therefore,

$$\int_C xy + z^2 \ ds = \frac{2\sqrt{8}}{3} + \frac{4\sqrt{8}}{3} + \frac{4\sqrt{8}}{3} = \frac{10\sqrt{8}}{3}.$$

Monday, November 11. We continued our discussion of line integrals of scalar function, working the following examples. The second of which illustrates how $\int_X f(x,y,z) \ ds$ is independent of the paramterization.

Example. Calculate $\int_C (x^2+y^2)z^3 \ ds$ for C that portion of the helix $\mathbf{r}(t) = (\cos(t), \sin(t), t)$, with $0 \le t \le \frac{\pi}{4}$.

Solution. We have $||\mathbf{r}'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} = \sqrt{2}$. Since $x = \cos(t), y = \sin(t), z = t$ along the curve,

$$\begin{aligned}
\int_C (x^2+y^2)z^3 \ ds &= \int_0^{\frac{\pi}{4}} (\cos^2(t) + \sin^2(t))t^3 \ \sqrt{2} \ dt \\
&= \sqrt{2} \int_0^{\frac{\pi}{4}} t^3 \ dt \\
&= \frac{\sqrt{2}}{4} \ t^4 \Big|_0^{\frac{\pi}{4}} \\
&= \frac{\sqrt{2}}{1024}\pi^4.
\end{aligned}$$

Example. Let C be the upper half of the unit circle in the xy-plane, centered at the origin. We verified that $\int_C f(x,y,z) \ ds$ is independent of the parametrization for $f(x,y) = ye^x$ and the parametrizations: $\mathbf{r}(t) = (\cos(t), \sin(t))$, $0 \le t \le \pi$ and $\mathbf{s}(t) = (\cos(\pi - 2t), \sin(\pi - 2t))$, $0 \le t \le \frac{\pi}{2}$. For $\mathbf{r}(t)$ we have

$$||\mathbf{r}'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1,$$

so that

$$\begin{aligned}
\int_C ye^x \ ds &= \int_0^\pi \sin(t)e^{cos(t)} \ dt \\
&= -e^{\cos(t)} \Big|_{t=0}^{t=\pi} \\
&= -e^{-1} + e.
\end{aligned}$$

For the parametrization $\mathbf{s}(t)$ we have

$$||\mathbf{s}'(t)|| = \sqrt{(2\cos(\pi - 2t))^2 + (-2\sin(\pi - 2t)^2} = 2,$$

so that,

$$\int_C ye^x \ ds = \int_0^{\frac{\pi}{2}} \sin(\pi - 2t)e^{\cos(\pi-2t)} \ 2 \ dt.$$

Using u-substitution with $u = \cos(\pi - 2t)$, we have

$$\begin{aligned}\int_C ye^x \ ds &= \int_0^{\frac{\pi}{2}} \sin(\pi - 2t)e^{\cos(\pi-2t)} \ 2 \ dt. \\ &= \int_{-1}^{1} e^u \ du \\ &= e - e^{-1},\end{aligned}$$

which agrees with the calculation above.

We then stated the following applications of a line integral.

Applications.

(i) $\frac{1}{\text{length}(C)} \int_c f(x,y,z) \ ds$ give the average value of $f(x,y,z)$ over C.

(ii) If $C \subseteq \mathbb{R}^2$ and $f(x,y) \geq 0$, for $(x,y) \in C$, then $\int_C f(x,y) \ ds$ represents the area under the surface $z = f(x,y)$ above the curve C.

(iii) If C represents a wire and $f(x,y,z)$ is the density of the wire at the point (x,y,z), then the *total mass* of the wire is $M = \int_C f(x,y,z) \ ds$.

(iv) The *center of mass* of the wire is the point $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{\int_C xf(x,y,z) \ ds}{M}, \quad \overline{y} = \frac{\int_C yf(x,y,z) \ ds}{M}, \quad \overline{z} = \frac{\int_C zf(x,y,z) \ ds}{M}.$$

We ended our initial discussion of line integrals by showing that if we are given two 1-1 parameterizations of the curve C, $\mathbf{r}(t)$, $a \leq t \leq b$ and $\mathbf{s}(t)$, $c \leq t \leq d$, with continuous derivatives, then

$$\int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||dt = \int_c^d f(\mathbf{s}(t))||\mathbf{s}'(t)||dt.$$

This was accomplished by viewing $\mathbf{s}(t)$ as a *re-parameterization* of $\mathbf{r}(t)$, i.e., by writing $\mathbf{s}(t) = \mathbf{r}(\theta(t))$, for $\theta : [c,d] \to [a,b]$, where θ is 1-1, with a continuous derivative, and $\theta(c) = a$ and $\theta(d) = b$.

We then began our discussion of surface integrals whose integrands are scalar functions. We follow the same process used in all previous forms of integration. To integrate $f(x,y,z)$ over the surface S,

Step 1. Subdivide S into finitely many smaller surfaces S_i of the same area ΔS. We are using ΔS for a small element of surface area.

Step 2. Choose a point (x_i, y_i, z_i) from the component S_i.

Step 3. Multiply $f(x_i, y_i, z_i)$ by the size of each S_i to get $f(x_i, y_i, z_i) \ \Delta S$.

Step 4. Add the products in Step 3 to get the Riemann sum: $\Sigma_i f(x_i, y_i, z_i)\Delta S$.

Step 5. Take the limit of the Riemann sums as $\Delta S \to 0$, to get:

$$\int\int_S f(x,y,z) \ dS,$$

the surface integral of $f(x,y,)$ over S. Note that we write a double integral, since our domain of integration is two-dimensional.

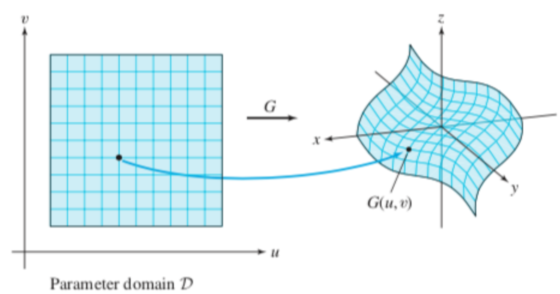
Following an analogy with curves, to calculate $\int\int_S f(x, y, z)\ dS$, we will need:

(i) A way to describe or *parametrize* a surface as a function of two variables.
(ii) A way to calculate surface area.

Definition. Given a surface $S \subseteq \mathbb{R}^3$, a parametrization of S will be a function

$$G(u, v) = (x(u, v), y(u, v), z(u, x)),$$

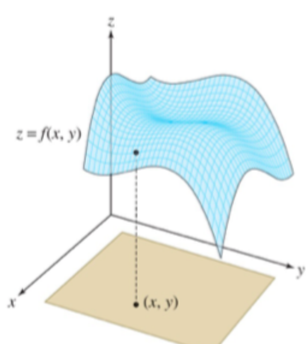
such that $S = G(D)$ for some domain D in the uv-plane.



As usual, we assume that all first order partials exists and are continuous, at least on the interior of D.

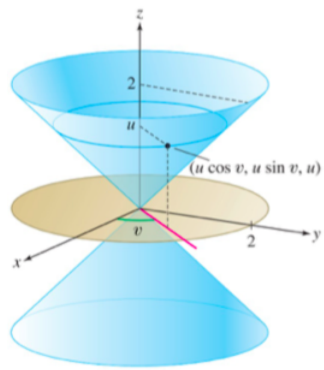
Wednesday, November 13. We continued our discussion of paramterizing surfaces by looking at the following examples.

Example 1. The easiest surface to parametrize is a surface that is the graph of $z = f(x, y)$. Why: Because it is already defined by two parameters!



Here we can write $G(u, v) = (u, v, f(u, v))$ or $G(x, y) = (x, y, f(x, y))$. For example, if S is that portion of the paraboloid $z = x^2+y^2$ lying over $D : 0 \le x^2+y^2 \le 9$, then $G(u, v) = (u, v, u^2+v^2)$, with $0 \le u^2+v^2 \le 9$ is a parametrization that takes the disk of radius 3 in the uv-plane to S.

Example 2. Consider the cone given by $z^2 = x^2 + y^2$, with $0 \le x^2 + y^2 \le 4$.



Though the top half of the cone can be expressed as $z = \sqrt{x^2+y^2}$ and we could parametrize it by $G(u,v) = (u, v, \sqrt{u^2+v^2})$ with $0 \le x^2+y^2 \le 4$, the whole surface cannot be expressed as the graph of a function of x and y.

Better: $G(u,v) = (u\cos(v), u\sin(v)u)$, with $-2 \le u \le 2$ and $0 \le v \le 2\pi$. Note

$$u^2 = (u\cos(v))^2 + (u\sin(v))^2,$$

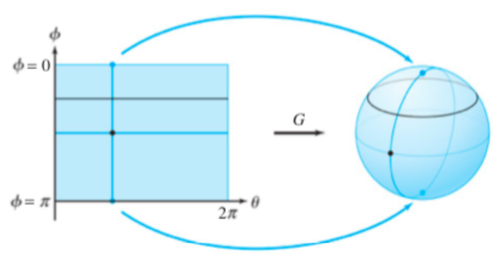
so the points $G(u,v)$ lie on the cone. More over, if we hold u fixed at u_0 and let v vary between 0 and 2π, this vertical line segment in the uv-plane is taken by G to the circle $G(u_0, v) = (u_0\cos(v), u_0\sin(v), u_0)$, i.e., the circle of radius u_0 centered at $(0,0,u_0)$. As u_0 varies between 2 and -2, these circles sweep out the cone.

Spherical and cylindrical coordinates tell us how to parametrize spheres and cylinders.

Example 3. For S the sphere of radius R centered at the origin we take:

$$G(\phi,\theta) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)),$$

with $0 \le \phi \le \pi, 0 \le \theta < 2\pi$.



As we have seen in our earlier discussion concerning spherical coordinates, this transformation take the vertical line segment $\phi = \phi_0$, with $0 \le \theta \le 2\pi$ in the $\phi\theta$-plane to the circle $(R\sin(\phi_0)\cos(\theta), R\sin(\phi_0)\cos(\theta), R\cos(\phi_0))$ of radius $R\sin(\phi_0)$ centered at $(0,0,R\cos(\phi_0))$. As ϕ_0 varies from 0 to π these circles sweep our the sphere.

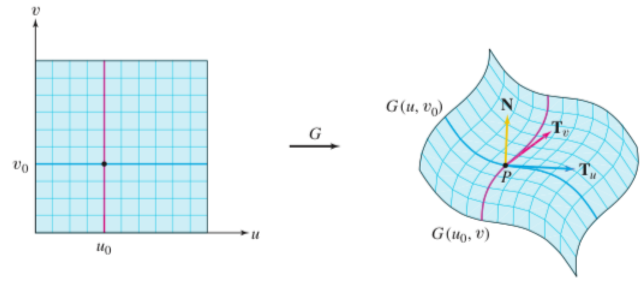
Now that we have a description of a surface in terms of parameters u, v, we can use this parameterization to find the plane tangent to the surface at $P = (x_0, y_0, z_0) = G(u_0, v_0)$. Since the equation of any plane is determined by a point on the plane and a vector normal to the plane, to find the tangent plane to the surface S at a point P, we find two tangent vectors, and take their cross product to find a normal vector.

Suppose $P = G(u_0, v_0)$. How do we get tangent vectors to S at P? Hold v fixed at v_0 and let u vary. Then $G(u, v_0)$ gives a curve C_1 that moves in the u-direction on S passing through P.
Thus, $\mathbf{T}_u(u_0, v_0) = \frac{\partial G}{\partial u}(u_0, v_0)$ is a vector tangent to C_1, and hence S, at P.

$$\mathbf{T}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)i + \frac{\partial y}{\partial u}(u_0, v_0)j + \frac{\partial z}{\partial u}(u_0, v_0)k.$$

We get a second tangent vector to S by taking a tangent to the curve obtained by holding u fixed at u_0.

$$\mathbf{T}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)i + \frac{\partial y}{\partial v}(u_0, v_0)j + \frac{\partial z}{\partial v}(u_0, v_0)k.$$



The normal vector to S at P is

$$\begin{aligned}\mathbf{N} =& \mathbf{T}_u(u_0, v_0) \times \mathbf{T}_v(u_0, v_0),\\ =& \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u}(u_0, v_0) & \frac{\partial y}{\partial u}(u_0, v_0) & \frac{\partial z}{\partial u}(u_0, v_0) \\ \frac{\partial x}{\partial v}(u_0, v_0) & \frac{\partial y}{\partial v}(u_0, v_0) & \frac{\partial z}{\partial v}(u_0, v_0) \end{vmatrix}.\end{aligned}$$

With this information, we can then find the equation of the plane tangent to S at the point $P = G(u_0, v_0)$.

Example 4. Find the equation of the tangent plane to the sphere of radius 2 centered at the origin at the point $P = (1, 1, \sqrt{2})$.

Solution. $G(\phi, \theta) = (2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi))$. $G(\frac{\pi}{4}, \frac{\pi}{4}) = (1, 1, \sqrt{2})$. We have:

$$\mathbf{T}_\phi = (2\cos(\phi)\cos(\theta), 2\cos(\phi)\sin(\theta), -2\sin(\phi)) \quad \text{and} \quad \mathbf{T}_\theta = (-2\sin(\phi)\sin(\theta), 2\sin(\phi)\cos(\theta), 0).$$

Thus, at the point P, we have

$$\mathbf{T}_\phi(\frac{\pi}{4}, \frac{\pi}{4}) = (1, 1, -\sqrt{2}) \quad \text{and} \quad \mathbf{T}_\theta(\frac{\pi}{4}, \frac{\pi}{4}) = (-1, 1, 0),$$

so

$$\mathbf{T}_\phi(\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbf{T}_\theta(\frac{\pi}{4}, \frac{\pi}{4}) = \begin{vmatrix} i & j & k \\ 1 & 1 & -\sqrt{2} \\ -1 & 1 & 0 \end{vmatrix} = \sqrt{2}i + \sqrt{2}j + 2k.$$

Therefore, the tangent plane at P is:

$$\sqrt{2}(x-1) + \sqrt{2}(y-1) + 2(z - \sqrt{2}) = 0.$$

We then noted that the process for finding a tangent plane to a parametrized surface a yields the same result for tangent planes given in the lecture of February 10. This follows, since if the surface is given by $z = f(x, y)$, then $G(x, y) = (x, y, f(x, y))$ is a parametrization. Thus, if we want the tangent vectors at the point $(a, b, f(a, b))$ using $G(x, y)$, we have $\mathbf{T}_x(a, b) = (1, 0, \frac{\partial f}{\partial x}(a, b))$ and $\mathbf{T}_y(a, b) = (0, 1, \frac{\partial f}{\partial y}(a, b))$, which are precisely the same tangent vectors as before.

Given the parametrized surface, why is $\mathbf{T}_u(u_0, v_0)$ tangent to the surface at $P = (x_0, y_0, z_0) = G(u_0, v_0)$? If we hold v fixed at v_0 then $\mathbf{r}(u) = (x(u, v_0), y(u, v_0), z(u, v_0))$ is a curve on the surface passing through P. Thus, $\mathbf{r}'(u_0)$ is tangent to the curve, and hence the surface, at $P = \mathbf{r}(u_0)$. However, $\mathbf{r}'(u_0) = \mathbf{T}_u(u_0, v_0)$, which shows the latter vector is tangent to the surface at P. Similarly, $\mathbf{T}_v(u_0, v_0)$ is also tangent to the surface at P. It follows that, as long as $\mathbf{N} = \mathbf{T}_u(u_0, v_0) \times \mathbf{T}_v(u_0, v_0) \neq \mathbf{0}$, we can use $\mathbf{N}$ to calculate the tangent plane to the surface at the point P.

Formula for surface area. If the bounded surface S is given by $G(u, v)$, with $G(D) = S$, for D in the uv-planes, then:

$$\text{surface area}(S) = \int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\, dA,$$

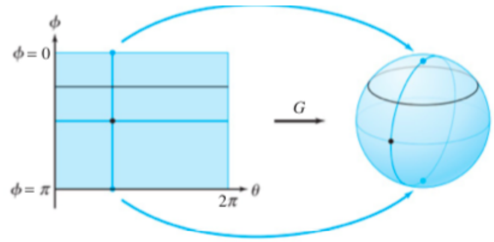
where the double integral is a standard double integral in the uv-plane.

Example 5. Find the surface area of a sphere S of radius R.

Solution. We use

$$G(\phi, \theta) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)),$$

with $0 \leq \phi \leq \pi, 0 \leq \theta < 2\pi$, which defines D.



$$\mathbf{T}_\phi = (R\cos(\phi)\cos(\theta), R\cos(\phi)\sin(\theta), -R\sin(\phi)) \quad \text{and} \quad \mathbf{T}_\theta = (-R\sin(\phi)\sin(\theta), R\sin(\phi)\cos(\theta), 0).$$

Thus,

$$\begin{aligned}
\mathbf{T}_\phi \times \mathbf{T}_\theta &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos(\phi)\cos(\theta) & R\cos(\phi)\sin(\theta) & -R\sin(\phi) \\ -R\sin(\phi)\sin(\theta) & R\sin(\phi)\cos(\theta) & 0 \end{vmatrix} \\
&= -R^2\sin(\phi)\{\sin(\phi)\cos(\theta)\vec{i} + \sin(\phi)\sin(\theta)\vec{j} + \cos(\phi)\vec{k}\}
\end{aligned}$$

We have:

$$||\mathbf{T}_\phi \times \mathbf{T}_\theta|| = R^2\sin(\phi),$$

since $(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$ lies on the sphere of radius 1. Note that $R^2\sin(\phi)$ is non-negative since $0 \leq \phi \leq \pi$. Thus:

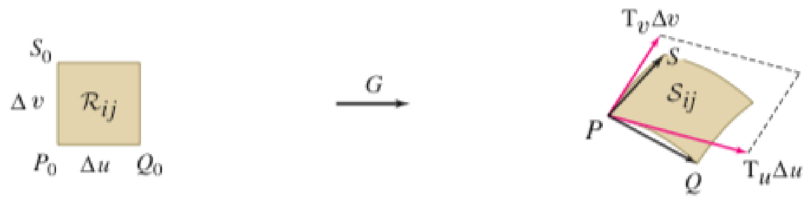
$$\begin{aligned}
\text{surface area}(S) &= \int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\ dA \\
&= \int_0^{2\pi}\int_0^{\pi} R^2\sin(\phi)\ d\phi d\theta \\
&= 2\pi R^2 \int_0^{\pi} \sin(\phi)\ d\phi \\
&= 2\pi R^2(-\cos(\phi))\Big|_0^{\pi} \\
&= 4\pi R^2.
\end{aligned}$$

Question: Do you see a connection between the formula for the volume of a sphere of radius R and its surface area?

Answer: The surface area is the derivative of the volume.

Friday, November 15. In order to understand our formulas for calculating surface area and surface integrals, we need to understand, where does the formula for surface area come from? For this we will use the fact from Calculus 2 that if $\mathbf{v}_1 = ai + bj + ck$ and $\mathbf{v}_2 = di + cj + ek$, then the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is $||\mathbf{v}_1 \times \mathbf{v}_2||$.

We first subdivide S into small portions, S_i with surface area ΔS. We approximate the small portions of surface area ΔS with small approximating tangent parallelograms. We start with the parametrization $G(u, v)$ of S. We use a tangent parallelogram to estimate the surface of the curved parallelogram on the surface.



Note that the vector from P to Q is $G(u_0 + \Delta u, v_0) - G(u_0, v_0)$. This vector is approximated by the tangent vector in red, $\Delta u \cdot \mathbf{T}_u(u_0, v_0)$. Similarly, the other tangent vector in red $\Delta v \cdot \mathbf{T}_v(u_0, v_0)$ approximates the vector $G(u_0, v_0 + \Delta v) - G(u_0, v_0)$. Thus, our small approximating tangent parallelogram is spanned by the vectors $\Delta u \cdot \mathbf{T}_u(u_0, v_0)$ and $\Delta v \cdot \mathbf{T}_v(u_0, v_0)$. The area of the approximating rectangle is

$$||(\Delta u \cdot \mathbf{T}_u) \times (\Delta v \cdot \mathbf{T}_v)|| = ||\mathbf{T_u}(u_0, v_0) \times \mathbf{T}_v(u_0, v_0)||\Delta u \Delta v.$$

Adding these areas for each component of the subdivision and taking the limit as $\Delta u, \Delta v \to 0$, gives a total surface area of $\int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\ dA$.

By retracing our steps in defining a surface integral, using a parameterization leads to a formula for calculating $\int\int_S f(x, y, z)\ dS$.

Step 1. Subdivide S into finitely many smaller surfaces S_i of area $\Delta S \approx ||\mathbf{T}_u \times \mathbf{T}_v||\ dA$.

Step 2. Choose a point $G(u_i, v_i)$ from the component S_i.

Step 3. Multiply $f(G(u_i, v_i))$ the area of S_i to get

$$f(G(u_i, v_i)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v)(u_i, v_i)||\ dA.$$

Step 4. Add the products in Step 3 to get the Riemann sum :

$$\Sigma_i f(G(u_i, v_i)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v)(u_i, v_i)||\ dA.$$

Step 5. Take the limit of the Riemann sums as $\Delta S \to 0$, to get:

$$\begin{aligned}\int\int_S f(x, y, z)\ dS &= \int\int_D f(G(u_i, v_i)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v||\ dA \\ &= \int\int_D f((x(u, v), y(u, v), z(u, v)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v||\ dudv.\end{aligned}$$

We then calculated the following examples.

Example. Calculate $\int\int_S \frac{1}{1+4(x^2+y^2)}\ dS$ for S the paraboloid $z = x^2 + y^2$, $0 \le z \le 4$.

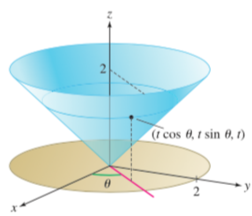
Solution. We take $G(u, v) = (u, v, u^2 + v^2)$, with $0 \le u^2 + v^2 \le 4$. We need to calculate $||\mathbf{T}_u \times \mathbf{T}_v||$.

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2ui - 2vj + k.$$

Thus $||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{(-2u)^2 + (-2v)^2 + 1^2} = \sqrt{4u^2 + 4v^2 + 1}$. Using this in the formula for surface integrals, we have

$$\begin{aligned}
\int\int_S \frac{1}{1+4(x^2+y^2)}\ dS &= \int\int_D \frac{1}{1+4(u^2+v^2)} \cdot \sqrt{1+4u^2+4v^2}\ dA \\
&= \int\int_D (1+4u^2+4v^2)^{-\frac{1}{2}}\ dudv \\
&= \int_0^{2\pi} \int_0^2 (1+4(r\cos(\theta))^2+4(r\sin(\theta))^2)^{-\frac{1}{2}}\ rdrd\theta \\
&= 2\pi \int_0^2 (1+4r^2)^{-\frac{1}{2}}\ rdr \\
&= 2\pi \cdot \frac{1}{4}(1+4r^2)^{\frac{1}{2}}\Big|_0^2 \\
&= \frac{\pi}{2}\{\sqrt{17}-1\}.
\end{aligned}$$

Example. Calculate $\int\int_S x^2 z\ dS$, where S is that portion of the cone $S : z^2 = x^2+y^2$ lying above the disk $D : 0 \leq x^2+y^2 \leq 4$.



Solution. From last lecture, $G(u, v) = (u\cos(v), u\sin(v), u)$, with $0 \leq u \leq 2$, $0 \leq v \leq 2\pi$, is a parametrization of S.

$\mathbf{T}_u = (\cos(v), \sin(v), 1)$ and $\mathbf{T}_v = (-u\sin(v), u\cos(v), 0)$.

$$\begin{aligned}
\mathbf{T}_u \times \mathbf{T}_v &= \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} \\
&= (-u\cos(v), -u\sin(v), u)
\end{aligned}$$

$||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{(-u\cos(v))^2+(-u\cos(v))^2+u^2} = \sqrt{2u^2} = \sqrt{2}|u|$.

$$\begin{aligned}
\int\int_S x^2 z\ dS &= \int\int_D (ucos(v))^2 u \cdot \sqrt{2}|u|\ dudv \\
&= \sqrt{2}\int_0^{2\pi}\int_0^2 u^4\cos^2(v)\ dudv,
\end{aligned}$$

since u is non-negative on D.

$$\begin{aligned}
&= \sqrt{2}\cdot\frac{32}{5}\int_0^{2\pi}\cos^2(v)\ dv \\
&= \frac{32\sqrt{2}}{5}\int_0^{2\pi}\frac{1}{2}+\frac{1}{2}\cos(2v)\ dv \\
&= \frac{32\sqrt{2}}{5}(\frac{v}{2}+\frac{1}{4}\sin(2v))\Big|_0^{2\pi} \\
&= \frac{32\sqrt{2}}{5}\pi
\end{aligned}$$

We then recorded the following familiar looking properties for surface integrals of continuous functions defined on a surface S:

(i) $\int \int_S f + g \ dS = \int \int_S f \ dS + \int \int_S g \ dS$.
(ii) $\int \int_S \lambda f \ dS = \lambda \int \int_S f \ dS$, for $\lambda \in \mathbb{R}$.
(iii) If $S = S_1 \cup S_2$, then $\int \int_S f \ dS = \int \int_{S_1} f] dS + \int \int_{S_2} f \ dS$, as long as S_1 and S_2 only intersect along their boundaries.
(iv) Surface area$(S) = \int \int_S \ dS$.

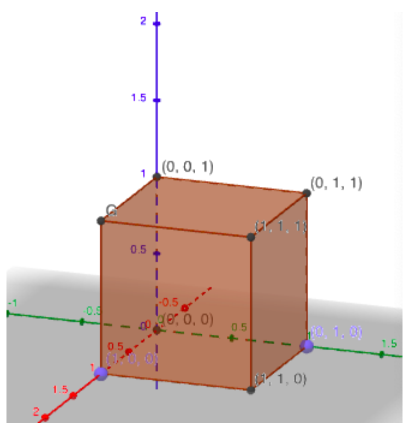
Important comments about calculating surface area and surface integrals.

1. The quantity $||\mathbf{T}_u \times \mathbf{T}_v||$ should always be non-negative (and almost never zero). If you you calculate this expression and it involves variables from the parametrization, this function should be positive on the domain of integration. For example, if you get $||\mathbf{T}_u \times \mathbf{T}_v|| = 2uv$, then you have to make sure that the product uv is positive on the domain of integration. If not, you may have dropped an absolute value somewhere.

2. Aside from having component functions with continuous first order partial derivatives, $G(u, v)$ should be 1-1 on the interior of the domain D. Otherwise, one may be double counting portions of the surface area.

3. For calculating surface area and surface integrals of scalar functions, the orientation of the normal vector does not matter, because these quantities use the length of the normal vector. However, it will matter when we integrate vector valued functions over a surface. After all, we care whether or not a fluid flows into or out of a chamber. Or as Hagrid might say: Better out than in.

We ended class by working the following example, showing how to use Property (iii) above.
Example. Calculate $\int \int_S xyz \ dS$, where S is the union of the six faces of the unit cube in $\mathbb{R}^3$.



Solution. Let $S_1, S_2, S_3, S_4, S_5, S_6$ respectively denote the bottom, left, back, right, front, and top faces of S, so that

$$\int \int_S xyz \ dS = \int \int_{S_1} xyz \ dS + \cdots + \int \int_{S_6} xyz \ dS.$$

Note that on $z = 0$ on S_1, and thus $xyz = 0$ on S_1, which gives $\int \int_{S_1} xyz \ dS = 0$. Similarly, $y = 0$ on S_2, so that $\int \int_{S_2} xyz \ dS = 0$ and $x = 0$ on S_3, so that $\int \int_{S_3} xyz \ dS = 0$.

We may parametrize the remaining sides of S as follows:

(i) $S_4 : G(u, v) = (u, 1, v)$ with $0 \leq u, v \leq 1$.
(ii) $S_5 : G(u, v) = (1, u, v)$, with $0 \leq u, v \leq 1$.
(iii) $S_6 : G(u, v) = (u, v, 1)$, with $0 \leq u, v \leq 1$.

Note that on each of these surfaces, $xyz = uv$. For S_4 : $\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -j$, so that $||\mathbf{T}_u \times \mathbf{T}_v|| = 1$.

Thus,

$$\begin{aligned}\int\int_{S_4} xyz\ dS &= \int_0^1 \int_0^1 uv\ 1 \cdot du\ dv \\ &= \frac{1}{2}\int_0^1 v\ dv \\ &= \frac{1}{2}\cdot\frac{1}{2} \\ &= \frac{1}{4}.\end{aligned}$$

It is easy to check that for S_5, $\mathbf{T}_u \times \mathbf{T}_v = i$, so that $||\mathbf{T}_u \times \mathbf{T}_v|| = 1$. Thus,

$$\int\int_{S_5} xyz\ dS = \int_0^1 \int_0^1 uv\ dv\ dv = \frac{1}{4}.$$

Likewise,

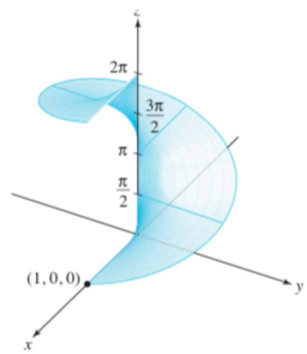
$$\int\int_{S_6} xyz\ dS = \int_0^1 \int_0^1 uv\ dv\ dv = \frac{1}{4}.$$

Thus,

$$\int\int_S xyz\ dS = 0 + 0 + 0 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Monday, November 18. We began class with the following example.

Example. Calculate $\int\int_S \sqrt{x^2+y^2+1}\ dS$ where S is the helicoid: $G(r,\theta) = (r\cos(\theta), r\sin(\theta), \theta)$, with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.



Solution. $\mathbf{T}_r = (\cos(\theta), \sin(\theta), 0)$ and $\mathbf{T}_\theta = (-r\sin(\theta), r\cos(\theta), 1)$.

$$\begin{aligned}\mathbf{T}_r \times \mathbf{T}_\theta &= \begin{vmatrix} i & j & k \\ \cos(\theta) & \sin(\theta) & 0 \\ -r\sin(\theta) & r\cos(\theta) & 1 \end{vmatrix} \\ &= (\sin(\theta), -\cos(\theta), r).\end{aligned}$$

$||\mathbf{T}_r \times \mathbf{T}_\theta|| = \sqrt{\sin^2(\theta) + (-\cos(\theta))^2 + r^2} = \sqrt{1+r^2}$.

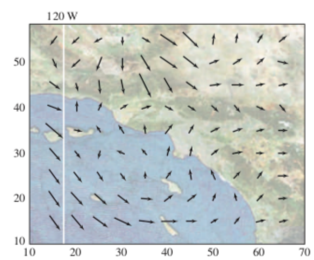
$$\int\int_S \sqrt{x^2+y^2+1}\, dS = \int_0^{2\pi}\int_0^1 \sqrt{(r\cos(\theta))^2+(r\sin(\theta))^2+1}\cdot\sqrt{r^2+1}\; drd\theta$$
$$= \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\cdot\sqrt{r^2+1}\; drd\theta$$
$$= 2\pi\int_0^1 r^2+1\; dr$$
$$= 2\pi\cdot\frac{4}{3}$$
$$= \frac{8\pi}{3}.$$

We then began a discussion of integrating vector fields over curves and surfaces.

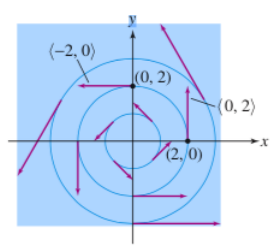
Definition. A vector field is a vector valued function

$$\mathbf{F}(x,y,z) = F_1(x,y,z)i + F_2(x,y,z,)j + F_3(x,y,z)k,$$

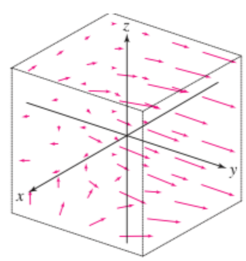
that assigns a vector to each point in a subset $S \subseteq \mathbb{R}^3$. Here's a picture of wind speed and wind direction as a vector field.



We will make our usual assumptions that the first order partial derivatives of all component functions exist and are continuous in suitable regions contained in $\mathbb{R}^3$. Some other pictures of vector fields:



$\mathbf{F} = \langle -y, x\rangle$



(A) $\mathbf{F} = \langle x\sin z, y^2, x/(z^2+1)\rangle$

Tangent vector fields and normals vector fields play a central role in what we want to do next.

Case 1. Let C be a smooth curve. Then at each point $P = (x,y,z)$ of the curve, we can assign a tangent vector $\mathbf{F}(x,y,z)$ to C at the point P. This gives a vector field along the curve C. Note that if $\mathbf{r}(t)$ is a parametrization of C, then $\mathbf{r}'(t)$ is a tangent vector at each point along the curve. Recall that $\mathbf{r}'(t)$ points in the direction of a point traveling along the curve and the length of $\mathbf{r}'(t)$ gives the speed of the point at time t.

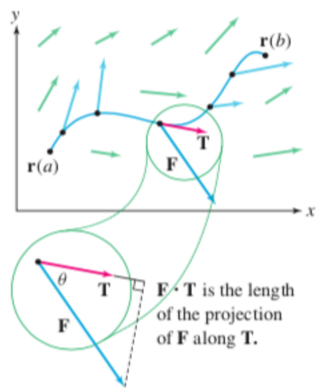
If we want to just keep track of the direction of a point moving along the curve, we can consider the unit tangent vector $\mathbf{T}(x,y,z)$ along the curve. If $\mathbf{r}(t)$ is a parametrization of C and $\mathbf{r}'(t)\neq 0$, then

$$\mathbf{T}(x,y,z) = \frac{1}{||\mathbf{r}'(t)||}\cdot\mathbf{r}'(t).$$

How do we use the unit tangent vectors? If we have a vector field $\mathbf{F}$ pushing a point along a curve, the dot product $\mathbf{F}(P) \cdot \mathbf{T}(P)$ gives us the component of $\mathbf{F}$ in the direction of the curve since:

$$\mathbf{F}(P) \cdot \mathbf{T}(P) = ||\mathbf{F}(P)|| \cdot ||\mathbf{T}(P)|| \cos(\theta) = ||\mathbf{F}(P)|| \cos(\theta),$$

where θ is the angle between $\mathbf{F}(P)$ and $\mathbf{T}(P)$.



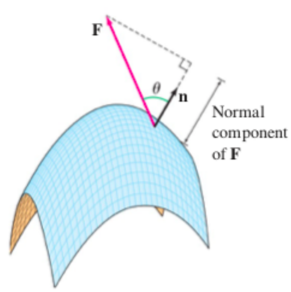
Informally, we can think of $\mathbf{F} \cdot \mathbf{T}$ as "how much of $\mathbf{F}$ acts in the direction of $\mathbf{T}$". We will see that when we want to integrate a vector field $\mathbf{F}$ along the curve C, we will really be integrating the scalar function $\mathbf{F} \cdot \mathbf{T}$ along the C.

Case 2. If we have a smooth surface S, then the normal vector $\mathbf{N}$ at each point of S defines a vector field on S.

Let S be a smooth surface in $\mathbb{R}^3$. We obtain a vector field $\mathbf{F}(x, y, z)$ by assigning to each point P on the surface a vector that is normal to S at P. At each point there are two normal vectors, each the negative of the other. The parametrization of the surface may not always yield the desired vector. If $G(u, v)$ is a parametrization of S, then $\mathbf{N}(u, v) = \mathbf{T}_u \times \mathbf{T}_v$ gives a normal vector and

$$\mathbf{n}(u, v) = \frac{1}{||\mathbf{T}_u(u, v) \times \mathbf{T}(u, v)_v||} \cdot \mathbf{N}(\mathbf{u}, \mathbf{v}),$$

gives a unit normal to the surface. If we have a vector field $\mathbf{F}$ passing through a surface - think of a fluid passing through a membrane - then $\mathbf{F}(P) \cdot \mathbf{n}(P)$ gives the component of $\mathbf{F}(P)$ in the direction of $\mathbf{n}(P)$. In fluid mechanics, one would call this the flux of the fluid across the boundary S at the point P.



We can now define line and surface integrals of vector fields.

Definition. Given a curve $C : \mathbf{r}(t)$ and a vector field $\mathbf{F}$, the line integral of $\mathbf{F}$ along C is the line integral of the scalar function $\mathbf{F} \cdot \mathbf{T}$, and is denoted $\int_C \mathbf{F} \cdot d\mathbf{r}$. Here $\mathbf{T}$ is the unit tangent along the curve pointing in the direction we travel along the curve. In other words,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds,$$

where the quantity on the right is the line integral along C of the scalar function $\mathbf{F}\cdot\mathbf{T}$. In terms of calculating the line integral, given the parametrization $\mathbf{r}(t)$ we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (\mathbf{F}\cdot\mathbf{T})\ ds \\ &= \int_a^b \{\mathbf{F}(\mathbf{r}(t))\cdot(\frac{1}{||\mathbf{r}'(t)||}\mathbf{r}'(t))\}\ ||\mathbf{r}'(t)||dt \\ &= \int_a^b \mathbf{F}(x(t),y(t),z(t))\cdot\mathbf{r}'(t)\ dt,\end{aligned}$$

since $\frac{1}{||\mathbf{r}'(t)||}$ cancels with $||\mathbf{r}'(t)||$. Thus, we do not have to calculate $||\mathbf{r}'(t)||$ even though it is implicit in the definition of $\int_C \mathbf{F}\cdot d\mathbf{r}$. Note that to calculate $\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t),y(t),z(t))$, everywhere x,y, or z appear in the formula for $\mathbf{F}$ we replace these by $x(t),y(t),z(t)$ given in the parametrization of C.

Example. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, for C given by $\mathbf{r}(t) = (t+1, e^t, t^2),\ \ 0\le t\le 2$ and $\mathbf{F} = zi + y^2j + xk$.

Solution. $\mathbf{F}(\mathbf{r}(t)) = t^2i + (e^t)^2j + (t+1)k$ and $\mathbf{r}'(t) = i + e^tj + 2tk$. Thus,

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = t^2 + e^{3t} + 2t(t+1) = e^{3t} + 2t + 3t^2.$$

Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 e^{3t} + 2t + 3t^2\ dt = \frac{1}{3}(e^6+35).$$

Definition. Given an surface S parametrized by $G(u,v) = (x(u,v), y(u,v), z(u,v))$, and a vector field $\mathbf{F}$, the surface integral of $\mathbf{F}$ over S is the surface integral of the scalar function $\mathbf{F}\cdot\mathbf{n}$, and is denoted $\int\int_S \mathbf{F}\cdot d\mathbf{S}$. Here, $\mathbf{n}$ is the unit normal on the surface determined by the parametrization, i.e., $\mathbf{n} = \frac{1}{||\mathbf{T}_u\times\mathbf{T}_v||}\mathbf{T}_u\times\mathbf{T}_v$

In other words, by definition,

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int_S (\mathbf{F}\cdot\mathbf{n})\ dS.$$

We calculate $\int\int_S \mathbf{F}\cdot d\mathbf{S}$ using a parametrization. If

$$G(u,v) = (x(u,v), y(u,v), z(u,v)),\ \text{for } (u,v)\in D$$

is a parametrization of S, the surface integral of $\mathbf{F}$ over S is given by

$$\begin{aligned}\int\int_S \mathbf{F}\cdot d\mathbf{S} &= \int\int_S \mathbf{F}\cdot\mathbf{n}\ dS \\ &= \int\int_D \mathbf{F}(G(u,v))\cdot\{\frac{1}{||\mathbf{T}_u\times\mathbf{T}_v||}\mathbf{T}_u\times\mathbf{T}_v\}\ ||\mathbf{T}_u\times\mathbf{T}_v||\ dA \\ &= \int\int_D \mathbf{F}(x(u,v),y(u,v),z(u,v))\cdot\mathbf{T}_u\times\mathbf{T}_v\ dA.\end{aligned}$$

Thus, in a similar vein to the line integral above, in calculating $\int\int_S \mathbf{F}\cdot d\mathbf{S}$, we do not need to calculate $||\mathbf{T}_u\times\mathbf{T}_v||$, even though this quantity is implicit in the definition of $\int\int_S \mathbf{F}\cdot d\mathbf{S}$. Note also, that ultimately, the calculation of $\int\int_S \mathbf{F}\cdot d\mathbf{S}$ reduces to the calculation of a double integral of a function of u and v over the flat region D in the uv-plane. Moreover, $\mathbf{F}(G(u,v))$ is obtained by replacing each occurrence of x,y,z in the definition of $\mathbf{F}$ by $x(u,v), y(u,v), z(u,v)$ respectively.

Example. Calculate $\int\int \mathbf{F}\cdot d\mathbf{S}$ for S the upper hemisphere of the sphere of radius R centered at the origin, with the standard parametrization, and $\mathbf{F} = zi + xj + k$.

Solution. We have

$$G(\phi,\theta) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)),$$

with $0\le\phi\le\frac{\pi}{2}, 0\le\theta<2\pi$, which defines D.

$$\mathbf{T}_\phi = (R\cos(\phi)\cos(\theta), R\cos(\phi)\sin(\theta), -R\sin(\phi))\quad \text{and}\quad \mathbf{T}_\theta = (-R\sin(\phi)\sin(\theta), R\sin(\phi)\cos(\theta), 0).$$

Thus,

$$
\begin{aligned}
\mathbf{T}_\phi \times \mathbf{T}_\theta &= \begin{vmatrix} i & j & k \\ R\cos(\phi)\cos(\theta) & R\cos(\phi)\sin(\theta) & -R\sin(\phi) \\ -R\sin(\phi)\sin(\theta) & R\sin(\phi)\cos(\theta) & 0 \end{vmatrix} \\
&= R^2\sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))
\end{aligned}
$$

On the other hand,

$$\mathbf{F}(G(u,v)) = (R\cos(\phi), R\sin(\phi)\cos(\theta), 1),$$

Thus,

$$
\begin{aligned}
\mathbf{F}(G(u,v) \cdot \mathbf{T}_u \times \mathbf{T}_v &= (R\cos(\phi), R\sin(\phi)\cos(\theta), 1) \cdot R^2\sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)) \\
&= R^2\sin(\phi)\{R\cos(\phi)\sin(\phi)\cos(\theta) + R\sin(\phi)\cos(\theta)\sin(\phi)\sin(\theta) + \cos(\phi)\} \\
&= R^3\sin(\phi)\{\cos(\phi)\sin(\phi)\cos(\theta) + \sin^2(\phi)\cos(\theta)\sin(\theta) + \cos(\phi)\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int\int_S \mathbf{F} \cdot d\mathbf{S} &= R^3 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos(\phi)\sin^2\phi\cos(\theta) + \sin^3(\phi)\cos(\theta)\sin(\theta) + \cos(\phi)\sin(\phi)\ d\phi\ d\theta \\
&= R^3 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \cos(\phi)\sin^2\phi\cos(\theta) + \sin^3(\phi)\cos(\theta)\sin(\theta) + \cos(\phi)\sin(\phi)\ d\theta\ d\phi \\
&= R^3 \int_0^{\frac{\pi}{2}} \cos(\phi)\sin^2(\phi)(-\sin(\theta))_{\theta=0}^{\theta=2\pi} + \sin^3(\phi)(\frac{1}{2}\sin^2(\theta))_{\theta=0}^{\theta=2\pi} + \cos(\phi)\sin(\phi)\theta|_{\theta=0}^{\theta=2\pi}\ d\phi \\
&= 2\pi R^3 \int_0^{\frac{\pi}{2}} \cos(\phi)\sin(\phi)\ d\phi \\
&= \pi R^3\sin^2(\phi)\Big|_0^{\phi=\frac{\pi}{2}} \\
&= \pi R^3.
\end{aligned}
$$

Wednesday, November 20. We began class by recalling and discussing the formulas presented in the last lecture for integrating vectors fields along a curve or surface.

Example. We then revisited the examples from November 6 of the circle of radius one centered at (0,0,1) and its four parametrizations. We calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ over each path for $\mathbf{F} = y\vec{i} - xvj$.

(i) $\mathbf{r}_1(t) = (\cos(t), \sin(t), 1)$, with $0 \leq t \leq 2\pi$ traces the curve once.
(ii) $\mathbf{r}_2(t) = (\cos(2t), \sin(2t), 1)$, with $0 \leq t \leq \pi$ traces once, but twice as fast.
(ii) $\mathbf{r}_3(t) = (\cos(t), \sin(t), 1)$, with $0 \leq t \leq 4\pi$, traces the curve twice, but at the same speed at $\mathbf{r}_1(t)$.
(iv) $\mathbf{r}_4(t) = (\cos(2\pi - t, \sin(2\pi - t), 1)$, traces the curve, once in reverse order.

We expected the answers to depend to some degree on the paramterizations. We easily saw,

For $\mathbf{r}_1(t)$: $\int_C \mathbf{F} \cdot d = -2\pi$.

For $\mathbf{r}_2(t)$: $\int_C \mathbf{F} \cdot d = -2\pi$.

For $\mathbf{r}_3(t)$: $\int_C \mathbf{F} \cdot d = -4\pi$.

For $\mathbf{r}_4(t)$: $\int_C \mathbf{F} \cdot d = 2\pi$.

We then looked at the following interesting example.

Example. For $\mathbf{F} = (3y+1)\vec{i} + 3x\vec{j}$, calculate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ nd $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, for the curves $C_1 : (\cos(t), \sin(t))$, $0 \leq t \leq \frac{\pi}{2}$ and $C_2 : (1-t, t)$, $0 \leq t \leq 1$ connecting the points $P = (1,0)$ and $Q = (0,1)$.

Solution. For C_1, we have

$$
\begin{aligned}
\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\frac{\pi}{2}} (3\sin(t)+1, 3\cos(t))\cdot(-\sin(t),\cos(t)\ dt \\
&= \int_0^{\frac{\pi}{2}} (-3\sin^2(t)-\sin(t)+3\cos^2(t))\ dt \\
&= \int_0^{\frac{\pi}{2}} 3(\cos^2(t)-\sin^2(t))-\sin(t)\ dt \\
&= \int_0^{\frac{\pi}{2}} 3\cos(2t)-\sin(t)\ dt \\
&= (\frac{3}{2}\sin(2t)+\cos(t))\Big|_0^{\frac{\pi}{2}} \\
&= -1.
\end{aligned}
$$

For C_2 we have

$$
\begin{aligned}
\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 (3t-1, 3(1-t))\cdot(-1,1)\ dt \\
&= \int_0^1 -6t+2\ dt \\
&= (-3t^2+2t)\Big|_0^1 \\
&= -1.
\end{aligned}
$$

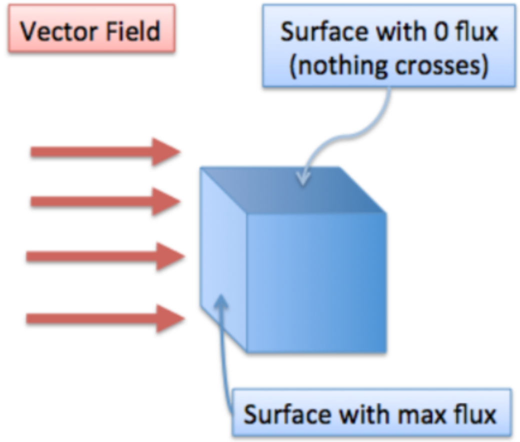
We noted that this example illustrated the concept of *path independence* enjoyed by certain vector fields $\mathbf{F}$, the so-called *conservative* vector fields. We noted that in this case $\mathbf{F} = \nabla f$, for $f(x, y) = 3xy + x$, and recorded, but did not discuss at length, the formula $\int_C \nabla f \cdot d\mathbf{r} = f(Q) - f(P)$, for the smooth curve C starting at the point P and ending at the point Q.

We ended class with a brief discussion of the concept of the *flux* of vector field.

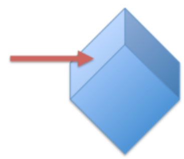
If we imagine that the vector field $\mathbf{F}$ represents a fluid passing through a membrane (e.g., blood through capillaries or an electromagnetic field passing though a shield) then, by definition, the flux is the quantity that passes through the membrane or shield. We want to see that the flux can be calculated as a surface integral.

Case 1. First consider a constant vector field $\mathbf{F}$ passing through a rectangle R perpendicular to the vectors in $\mathbf{F}$. How much of $\mathbf{F}$ passes through R? Answer:

$$
||\mathbf{F}||\cdot \text{area}(R) = (\mathbf{F}\cdot \frac{1}{||\mathbf{F}||}\mathbf{F})\cdot \text{area}(R) = (\mathbf{F}\cdot \mathbf{n})\cdot \text{area}(R).
$$



Note when $\mathbf{F}$ is parallel to R, the flux is zero. In this case $\mathbf{F} \cdot \mathbf{n} = 0 =$ flux. Case 2. $\mathbf{F}$ is constant, passing through R, neither perpendicular or parallel to R.

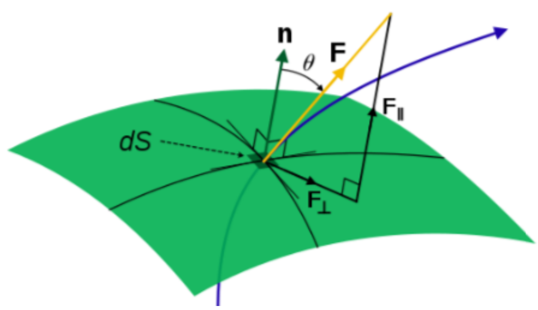


The amount of $\mathbf{F}$ flowing though R is $(\mathbf{F} \cdot \mathbf{n}) \cdot \text{area}(R)$.

General case. $\mathbf{F}$ is not constant, and the surface S is not a rectangle. Subdivide S into very small sections S_i of size dS. For $P_i \in S_i$, $\mathbf{F}$ on S_i is approximately $\mathbf{F}(P_i)$. The flux across each S_i is approximately: $\mathbf{F}(P_i) \cdot \mathbf{n}(P_i)\ dS$. Adding over the surface gives a Riemann sum:

$$\sum_i \mathbf{F}(P_I) \cdot \mathbf{n}(P_i)\ dS.$$

Taking the limit as $dS \to 0$ gives $\int \int_S \mathbf{F} \cdot \mathbf{n}\ dS = \int \int_S \mathbf{F} \cdot d\mathbf{S}$.



$$\text{Flux} = \int \int_S \mathbf{F} \cdot \mathbf{n}\ dS.$$